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AN "ADMISSIBLE" GENERALIZATION OF A THEOREM ON COUNTABLE Σ_1^1 SETS OF REALS WITH APPLICATIONS

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0. Introduction

A well-known result of J. Harrison [9] implies that any countable Σ_1^1 set of reals consists entirely of hyperarithmetic reals. We prove a generalization of this result, Theorem 1.5 below (our Main Lemma) which roughly says that for a given countable admissible set A, if T is a Σ-definable theory in L_A containing ε, if the axiom of extensionality is in T, if for all $\mathcal{M} \models$ T, the interpretation of the fixed constant c, $c^{\mathcal{M}}$, belongs to the standard part of $\mathcal{M}$, and finally, if the set of all $a \in$ HC which appear as $c^{\mathcal{M}}$ for some 'normal' $\mathcal{M} \models$ T ($\mathcal{M}$ normal meaning that $\varepsilon^{\mathcal{M}}$ on the standard part of $\mathcal{M}$ is identical to the real ε) is 'small', e.g., of power $< 2^{\aleph_0}$, then each $c^{\mathcal{M}}$ for 'normal' $\mathcal{M} \models$ T belongs to A. The proof relies on a refinement of Harrison's theorem which we derive from an improvement given by J. Barwise of a definability result of ours.

Also in Section 1, we apply the Main Lemma to give a proof of the Mansfield–Jensen theorem on Σ-subsets of HC without perfect subsets.

In Section 2, we give some preliminaries on canonical Scott sentences. Here we prove the apparently new result that the notion "ϕ is in A and ϕ is the canonical Scott sentence of some structure" is Δ on any admissible $A \subset$ HC containing ω, in fact, uniformly.

In Section 3, we apply the main lemma to obtain results concerning the canonical Scott sentences of models in a $PC_{\omega_1\omega}$ class containing 'few' countable models. Our main result here, Theorem 3.20, is a more precise version of Morley's theorem on the number of countable models.

In Section 4 we combine a result of Section 3 with recent work of J.-P. Ressayre. One of our results here, Corollary 4.8, says that if ϕ is a sentence of $L_{\omega_1\omega}$ categorical in $\aleph_1$, then the model of power $\aleph_1$ of ϕ is ∞, ω-equivalent to a countable model.

The first stage of the work presented here was announced in [14]. This included a proof of Mansfield's theorem on perfect subsets of Σ-subsets of HC (cf. 1.6 below) and a proof of Nadel's theorem on canonical Scott sentences (cf. 3.14 below). At the time we thought that we had the only forcing free proof of Mansfield's theorem. However, at that time we had not had the Main Lemma 1.4 (or 1.5) in full strength,

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only for admissible sets satisfying a stronger separation axiom. Then we received the preprints [22], [23] and [18], the last giving a forcing free proof of Mansfield's theorem, and the communication of Professor Sacks' announcing his theorem, 3.3 below. The latter information prompted us to realize that our earlier proof with a slight change in fact established 1.4 and 3.8 and 3.10 in their full strength, giving Sacks' theorem as a corollary. We announced these findings in [15].

Some time after this we learned that Professor Sacks had a version of our result on canonical Scott sentences, 3.10 (and 3.20) although with a different notion of Scott sentence. This and other results of Sacks' will appear in [25].

It will be apparent to the reader that our work was deeply influenced by Mark Nadel's work [21–23].

Finally, we would like to express our thanks to Victor Harnik for many stimulating conversations, sometimes under adverse circumstances, concerning the material of the present paper.

1. The Main Lemma

We are going to state a purely modeltheoretic theorem on definability, Theorem 1.1. below, which is in fact an infinitary generalization of the definability theorem of [3] and [12]. Theorem 1.1, or rather, its Corollary 1.2, will be the main tool in the proof of our Main Lemma (1.4, 1.5). For the case $L' = L$, Theorem 1.1 was given in [13] as Theorem 1. Subsequently, J. Barwise has noticed the present more general version which is the useful formulation for applications. (For a very detailed presentation of Theorem 1.1, see Theorem 4.6 in Chapter IV of [2]). Actually, the present version is even slightly more general than Theorem IV.4.6 in [2] by having a Σ-definable theory instead of a single sentence of L_A, but the proof of it requires no essential change with respect to that in [2], which in turn requires no essential change with respect to the proof sketched in [13]. (Also, in a rather roundabout way using the quantifier "there are uncountably many", the version with a Σ-definable theory can be deduced from the version with a single sentence.)

First, some terminology. A *Σ_1^1 sentence over* $L_{\omega_1\omega}$ or simply, a *Σ_1^1 sentence*, is one of the form $\exists\bar{R}\phi$ with $\phi \in L'_{\omega_1\omega}$ with some language $L' \supset L$. $\bar{R}$ denotes the difference-set $L' - L$. Accordingly, an L-structure $\mathcal{M}$ satisfies $\exists\bar{R}\phi$ iff it has an L′-expansion satisfying ϕ. The notion of a *Π_1^1 sentence* $\forall\bar{R}\phi$ has an obvious dual meaning.

We will distinguish between Σ- and $\mathbf{\Sigma}$-definability on an admissible A in the usual way, the first meaning definability by a Σ-formula of set-theory without parameter, the second meaning the same but using parameters in A. Similarly for Π and Δ.

All languages in this paper are countable.

For Theorem 1.1 to be stated next, and for the most part of the paper, A denotes a countable admissible set, $L \subset L'$ are languages (sets of non-logical symbols), both

L and L′ are Δ-definable subsets of A. Furthermore, P is a further predicate symbol, $P \not\in$ L′ and Σ is a theory in the language $\mathrm{L}'_A(P)$. We will be interested in models $(\mathcal{M}, P)$ of the Σ^1_1 sentence $\exists \bar{R} \wedge \Sigma$, with $\bar{R} = \mathrm{L}' - \mathrm{L}$.

For a given L-structure $\mathcal{M}$, and any sentence σ, we denote by $P(\mathcal{M}, \sigma)$ or more simply by $P(\mathcal{M})$ the set of all predicates P on $|\mathcal{M}|$, the underlying set of $\mathcal{M}$, such that $(\mathcal{M}, P) \models \sigma$. We consider ${}^{n|\mathcal{M}|}2$, the set of all n-ary predicates on $|\mathcal{M}|$, a topological space with the usual product topology. When below we speak of a perfect subset of $P(\mathcal{M})$, we mean a non-empty subset of $P(\mathcal{M})$ which is closed in the space ${}^{n|\mathcal{M}|}2$ and has no isolated points.

The theorem is formulated as follows:

Theorem 1.1. *Let Σ be a theory as before and assume that Σ is Σ-definable on A. Let $\sigma = \exists \bar{R} \wedge \Sigma$. Then the following three conditions are equivalent.*

(i) *For every countable L-structure $\mathcal{M}$, $P(\mathcal{M}, \sigma)$ is countable (i.e., of power $\leqslant \aleph_0$).*

(ii) *For every $\mathcal{M}$ as in (i), $P(\mathcal{M}, \sigma)$ does not contain a perfect subset.*

(iii) *There is an A-finite set $\{\phi_i(x, u_1, \ldots, u_{m_i}) : i \in I\}$ of L_A-formulas such that*

$$\Sigma \models \bigvee_{i \in I} \exists u_1 \cdots u_{m_i} \forall x \, [Px \leftrightarrow \phi_i(x, u_1, \ldots, u_{m_i})].$$

Remark. There are generalizations of Theorem 1.1 and many of the results below in which "relativized" versions of the notions of admissibility and Σ-definability come in. For example, Theorems 1.1, 1.4, e.t.c., hold in the version where A is assumed to be admissible in some predicates $R_1, \ldots, R_n$ on A and Σ (resp., T) is assumed to be Σ-definable on A relative to $R_1, \ldots, R_n$. There is no new feature in the proofs of these generalizations. Although we have not found applications for such generalizations, they may very well turn out to be useful as suggested e.g. by G. Sacks' work (to appear in [25]) using such 'higher' admissibilities for model-theoretical purposes.

We will use the following terminology. Let C be an element of A. We call a set X of subsets of C (or of binary relations, e.t.c., on C) Σ^1_1 *relative* to A, or simply Σ^1_1 if A is implicit by the context, if X is of the form $P(\mathcal{M}, \sigma)$ for some $\sigma = \exists \bar{R} \wedge \Sigma$ with Σ a theory Σ-definable on A, and for some model $\mathcal{M}$ such that $|\mathcal{M}| = C$ and the (ordinary) diagram of $\mathcal{M}$, $\mathrm{diag}(\mathcal{M})$, is a Δ-subset of A (if the language of σ is A-finite, $\mathcal{M}$ is A-finite too).

For the purposes of the present paper, the following corollary of Theorem 1.1 will be sufficient.

Corollary 1.2. *Let $C \in A$, A be countable admissible and let X be a set of subsets of C which is Σ^1_1 relative to A. Suppose that X does not contain a perfect subset. Then $X \subset A$, in fact X is a subset of an A-finite set.*

Proof. Take $\exists \bar{R} \, [\wedge \mathrm{diag} \, \mathcal{M} \wedge \forall x \bigvee \{x = a : a \in |\mathcal{M}|\} \wedge \wedge \Sigma]$ where $\mathcal{M}$ and Σ are given by the definition of "Σ^1_1", as a new Σ^1_1 sentence $\exists \bar{R} \phi$ over the language

$L_{\mathcal{M}} = L \cup \{\mathbf{a}: a \in |\mathcal{M}|\}$. (Here diag $\mathcal{M}$ is the usual diagram of $\mathcal{M}$ utilizing constants $\mathbf{a}$ denoting $a \in |\mathcal{M}|$.) Theorem 1.1. applied to $\exists \bar{R}\phi$ gives Corollary 1.2.

Corollary 1.2 generalizes the following well-known result due to J. Harrison [9], cf. also [17]:

(*) Let X be a set of reals, let X be Σ_1^1 in the usual sense in the fixed real a. If X does not contain a perfect subset, then each element of X is hyperarithmetic in a.

(*) in turn is a more precise version of the classical theorem of descriptive set theory saying that an uncountable analytic set contains a perfect subset.

Using forcing and the Barwise Σ-compactness theorem, one can derive Corollary 1.2 from (*). This fact sheds light on the use of forcing in earlier proofs of Mansfield's theorem on Σ-subsets of HC (we note that Mansfield gives a forcing-free proof of his theorem in [18].

Definition 1.3. Let A be a set, possibly containing some urelements. Let X be a class of sets, (or, in case A has a subset U of urelements, let X be a subclass of V_U (cf. [2])). We call X *A-scattered* if for any $B \in A$, $\mathcal{P}(B) \cap \mathrm{TC}(X)$ does not contain a perfect subset (perfect is understood relative to the ordinary product topology on ${}^B2$; $\mathcal{P}(B)$ is the power set of B, $\mathrm{TC}(X)$ is the transitive closure of X). X is *scattered* if it is HC-scattered (HC is the set of hereditarily countable sets).

Remark. If $|X| < 2^{\aleph_0}$, then X is obviously scattered.

We note that for any *extensional* structure $\mathcal{M} = (|\mathcal{M}|, \in^{\mathcal{M}}, \ldots,)$ i.e. one that satisfies ext, the axiom of extensionality, the standard part of $\mathcal{M}$, $\mathrm{WF}(\mathcal{M})$, can be defined as the smallest set $X \subset |\mathcal{M}|$ such that if $x \in |\mathcal{M}|$ and $\{y: y \in^{\mathcal{M}} x\} \subset X$ then $x \in X$, and $(\mathrm{WF}(\mathcal{M}), \in^{\mathcal{M}} \restriction \mathrm{WF}(\mathcal{M}))$ will be isomorphic to $(w, \in \restriction w)$ for some transitive set w. Also, $\mathrm{WF}(\mathcal{M})$ can be defined as the set of $x \in |\mathcal{M}|$ such that there is no $\in^{\mathcal{M}}$-descending sequences $x = x_0 \ni^{\mathcal{M}} x_1 \ni^{\mathcal{M}} x_2 \ni^{\mathcal{M}} \cdots$. An extensional $\mathcal{M}$ is called *normal* if for $x, y \in \mathrm{WF}(\mathcal{M})$, $x \in^{\mathcal{M}} y \Leftrightarrow x \in y$. Every extensional structure is isomorphic to a normal one.

Again, let A be a fixed countable admissible set, $\omega \in A$ (instead of $\omega \in A$, we could require that A is an admissible set with urelements (cf. [2]), and that the set of urelements is infinite and it belongs to A). Let T be a theory in a language L_A containing $\in$ and the constant c; let L be Δ-definable on A, T Σ-definable on A. Assume that $\mathrm{KP} \subseteq \mathrm{T}$ (KP is the Kripke–Platek axiom system, cf. e.g. [2]) and that every model of T is an end extension of $(A, \in \restriction A)$.

Theorem 1.4 (Main Lemma, first version). *In addition to the above conditions on* T, *assume that*

(i) *for every* $\mathcal{M} \models \mathrm{T}$, $c^{\mathcal{M}} \in \mathrm{WF}(\mathcal{M})$,

and

(ii) *the set* $\mathrm{tr}(\mathrm{T}) =_{\mathrm{df}} \{c^{\mathcal{M}} : \mathcal{M}$ *is a normal model of* $\mathrm{T}\}$ *is A-scattered.*

Then $\mathrm{tr}(\mathrm{T})$ *is a subset of an A-finite set.*

Proof. For each $a \in A$, let $\boldsymbol{a}$ be a distinct new individual constant and let diag(A) to be usual *infinitary* diagram of $(A, \in \restriction A)$ whose normal models $\mathcal{M}$ are exactly the endextensions of $(A, \in \restriction A)$ with $\boldsymbol{a}^{\mathcal{M}} = a$. We can assume that diag$(A) \subset \mathrm{T}$ by passing to $\mathrm{T} \cup \mathrm{diag}(A)$ if necessary.

First we claim that under the hypotheses of the theorem, there is $\alpha < o(A)$ ($=$ smallest ordinal not in A) such that $r(x) < \alpha$ for each $x \in \mathrm{tr}(\mathrm{T})$. ($r(x)$ is the set-theoretic rank of x). Assume this is false and consider the theory T′ defined as

$$\mathrm{T} \cup \{\text{“}r(\mathrm{c}) > \boldsymbol{\alpha}\text{”}: \alpha < o(A)\}.$$

T′ is A-finitely consistent by assumption, hence by the Barwise compactness theorem it has a model. Recall the theorem of H. Friedman [5] according to which T′ has a model $\mathcal{M}$ such that the set of standard ordinals in $\mathcal{M}$ is precisely $o(A)$. Consider such an $\mathcal{M}$. Clearly, $c^{\mathcal{M}} \notin \mathrm{WF}(\mathcal{M})$, a contradiction to (i). The claim is shown.

Next we show that we can assume that tr(T) is transitive. Given T, we pass to T_1 defined by

$$\mathrm{T}_1 = \mathrm{T} \cup \{\text{“}d \in \{c\} \cup \mathrm{TC}(c)\text{”}\}.$$

Then $\mathrm{tr}(\mathrm{T}_1) =_{\mathrm{df}} \{d^{\mathcal{M}'} : \mathcal{M}' \models \mathrm{T}_1,\ \mathcal{M}' \text{ is normal}\}$ coincides with TC(tr(T)), clearly establishing the claim.

Now assume, in addition, that tr(T) is transitive. For arbitrary ordinals α, define

$$X_\alpha =_{\mathrm{df}} \{a \in \mathrm{tr}(\mathrm{T}) : r(a) \leq \alpha\}.$$

We show by induction on α that for each $\alpha < o(A)$, $(*)$ X_α is a subset of an A-finite set.

For $\alpha = 0$, and in the case there are urelements, $(*)$ follows from the assumption that the urelements form an A-finite set. Assume $0 < \alpha < o(A)$ and that $(*)$ is true for $\beta < \alpha$.

First we show that $X'_\alpha =_{\mathrm{df}} \bigcup_{\beta<\alpha} X_\beta$ is a subset of some A-finite set w. We can express the induction hypothesis as follows:

$$(A, \in \restriction A) \models \forall \beta < \alpha\ \exists x \text{ “}\mathrm{T} \cup \{\text{“}r(c) \leq \boldsymbol{\beta}\text{”}\} \models c \in \boldsymbol{x}\text{”}.$$

The predicate between the outer “ ” is a Σ-predicate of x and β (by the Barwise completeness–compactness theorem), hence we can apply Σ-collection on A to get a transitive $w \in A$ such that

$$(A, \in \restriction A) \models \forall \beta < \alpha\ \exists x \subset w \text{ “}\mathrm{T} \cup \{\text{“}r(c) \leq \boldsymbol{\beta}\text{”}\} \models c \in \boldsymbol{x}\text{”}.$$

This implies that $X'_\alpha \subset w$ as claimed.

By applying Δ-separation, we can further assume that every element of w is of rank $\leq \alpha$.

Next notice that $X_\alpha = \{x \subset X'_\alpha$: there is a normal model $\mathcal{M}$ of T such that $x = c^{\mathcal{M}}\} = \{x \subset w$: there is a normal $\mathcal{M} \models \mathrm{T}$ such that $x = c^{\mathcal{M}}\}$. We want to conclude that X_α is a Σ_1^1 subset of w (in the sense made precise before Corollary 1.2 in Section 1), but first we need to eliminate the word ‘normal’.

We have:

$x \in X_\alpha \Leftrightarrow x \subset w$ and there is $\mathcal{M} \models \mathrm{T}$, $\mathcal{M} = (|\mathrm{M}|, \in^{\mathcal{M}}, c^{\mathcal{M}}, a^{\mathcal{M}}, \ldots)_{a \in A}$ such that for every $a \in w$, $a \in x$ iff $a^{\mathcal{M}} \in^{\mathcal{M}} c^{\mathcal{M}}$.

Notice that w can be chosen infinite, hence $|\mathcal{M}|$ can be assumed to be a subset of w. Thus, X_α is of the form $P(\circledast, \exists \bar{R} \wedge \mathrm{T}_1)$ where $\circledast = (w, \in \restriction w)$ and T_1 is some appropriate translation of the right-hand side of the last equivalence, with P denoting x. In other words, X_α is a Σ_1^1 subset of w relative to A. Hence, we can apply hypothesis (ii) of the Theorem and Corollary 1.2 to conclude that X_α is a subset of an A-finite set, completing the inductive proof of $(*)$.

We have already noted that $\mathrm{tr(T)} \subset X_\alpha$ for some $\alpha < o(A)$. The proof is complete.

The following is a slight generalization of 1.4, obtained by relaxing the conditions on T a bit.

Theorem 1.5 (Main Lemma, second version). *Assume that* T *is a* Σ*-definable theory in* L_A, L *contains* ε *and* c *as before. Assume that ext belongs to* T. *Then, if* (i) *and* (ii) *of* 1.4 *are true, then the same conclusion holds as in* 1.4.

Proof. We will reduce 1.5 to 1.4. Introduce the new unary predicates $\boldsymbol{M}$ and $\boldsymbol{W}$ (to denote the universe of a model of T and its standard part, resp.), the binary predicate E (to replace $\in$ in T but also we use $\in$ as a formal predicate as indicated below). The axioms of a new theory T′ are as follows:

$$\begin{array}{l} \mathrm{KP} \\ \mathrm{diag}(A) \\ \phi^{(M,E)} \quad \text{for} \quad \phi \in \mathrm{T} \end{array}$$

where $\phi^{(M,E)}$ denotes the sentence obtained from ϕ by relativizing all quantifiers to $\boldsymbol{M}$ and replacing $\in$ by E, (we assume that the language of T did not contain operation symbols or individual constants other than c)

"$\boldsymbol{W}$ is an E-transitive subclass of $\boldsymbol{M}$",

"$\boldsymbol{W}$ is $\in$-transitive",

"$\boldsymbol{E}$ and $\in$ coincide on $\boldsymbol{W}$",

$\boldsymbol{W}(c)$.

Using assumption (i) on T, we can see that assumption (i) holds for T′ too and that tr(T′) = tr(T). When checking tr(T) ⊂ tr(T′), we take the normal $\mathcal{M} \models \mathrm{T}$, turn $\in^{\mathcal{M}}$ into E, define $\boldsymbol{W}$ as $\mathrm{WF}(\mathcal{M})$, extend $\boldsymbol{W} \cup A$ to some admissible set A' which will be the universe of the new model $\mathcal{M}'$ of T′, and finally project $|\mathcal{M}|$ onto some $\boldsymbol{M} \subset A'$ with keeping the projection the identity on $\boldsymbol{W}$. If the structure thus obtained, suitable for the language of T′, is $\mathcal{M}'$, then clearly, $c^{\mathcal{M}'} = c^{\mathcal{M}}$ and $\mathcal{M}' \models \mathrm{T}'$.

It is obvious that T′ satisfies the more restrictive conditions of 2.3. Q.E.D.

Let ω_1 denote the *real* first uncountable ordinal (as opposed to $\omega_1^{\mathscr{L}}$, the first constructively uncountable one) and let $\mathscr{L}_{\omega_1}[b]$ be the class of sets constructible from b below ω_1. Clearly, for $b \in \mathrm{HC}$, $\mathscr{L}_{\omega_1}[b] \subset \mathrm{HC}$. The proof of the next theorem given below was obtained independently of the recent [18] and knowing only a proof using forcing. The present proof is perhaps of interest even having the proof of the first part of 1.C in [18].

Theorem 1.6. *Suppose that X is a Σ-definable subset of* HC *from the parameter b and that it is $\mathscr{L}_{\omega_1}[b]$-scattered. Then $X \subset \mathscr{L}_{\omega_1}[b]$* (Mansfield [18]) *and in fact, X is a Σ-definable subset of $\mathscr{L}_{\omega_1}[b]$* (R. Jensen).

Proof. There is a Σ-formula $\phi(x)$ of set-theory using the parameter b such that for any $a \in \mathrm{HC}$,

$$a \in X \Leftrightarrow (\mathrm{HC}, \in \restriction \mathrm{HC})_{\mathrm{df}} = \mathrm{HC} \models \phi[a].$$

By the downward Löwenheim–Skolem theorem and by the fact that ϕ is Σ,

$a \in X \Leftrightarrow$ there is a standard transitive

$\circledast = (w, \in \restriction w)$ such that

$b \in w$, $a \in w$ and $\circledast \models \phi[a]$.

In case $\mathcal{M} = (|\mathcal{M}|, \in^{\mathcal{M}}, \ldots)$ satisfies KP, $\mathcal{M}$ has only standard ordinals and $\mathcal{M}$ is *normal*, then $\mathcal{M}$ is obviously a transitive $\in$-model. For any $\alpha < \omega_1$, define $a \in X_\alpha \Leftrightarrow$ there is a normal model $\mathcal{M} = (|\mathcal{M}|, \in^{\mathcal{M}}, c)$ of the following:

KP,

"$\langle \mathrm{ord}(\mathcal{M}), \in^{\mathcal{M}} \restriction \mathrm{ord}(\mathcal{M})\rangle$ is similar to $\langle \alpha, \in \restriction \alpha\rangle$"

$\phi(c)$

such that $a = c^{\mathcal{M}}$.

Then we have $X = \bigcup_{\alpha<\omega_1} X_\alpha$.

Fix $\alpha < \omega_1$. By assumption X_α is A-scattered for A the smallest admissible set such that $\alpha \in A$ and $b \in A$, since $A \subset \mathscr{L}_{\omega_1}[b]$. Clearly, X_α is of the form tr(T) for the theory T displayed in the definition of X_α. Also, T satisfies the requirements of 1.5. Thus, $X_\alpha \subset A$, hence $X = \bigcup_{\alpha<\omega_1} X_\alpha \subset \mathscr{L}_{\omega_1}[b]$.

To prove the second part of the theorem, recall the following result of Vaught [26]. Given any Σ_1^1 sentence $\exists \bar{R}\phi$ over $\mathrm{L}_{\omega_1\omega}$, there is a sequence of sentences δ_β in $\mathrm{L}_{\omega_1\omega}$ $\langle \delta_\beta \colon \beta < \omega_1\rangle$, such that δ_β in fact is a set-primitive recursive function of $\exists \bar{R}\phi$ and β, and for any L-structure $\mathcal{M}$ and any admissible set A with $\mathcal{M} \in A$ and $\exists \bar{R}\phi \in A$, we have that

$$\mathcal{M} \models \exists \bar{R}\phi \Leftrightarrow \bigwedge_{\beta<o(A)} \mathcal{M} \models \delta_\beta \Leftrightarrow \bigwedge_{\beta<\omega_1} \mathcal{M} \models \delta_\beta.$$

For any $a \in \mathrm{HC}$, denote the structure $(w, \in \restriction w)$ for $w = \{a\} \cup \mathrm{TC}(a)$ by ⓐ. a is the unique $\in$-maximal element of ⓐ.

Returning to the definition of X_α, we clearly have a Σ_1^1 sentence $\exists \vec{R}\phi_\alpha$ over the language $\{\in\}$ such that for any $a \in \mathrm{HC}$,

$$a \in X_\alpha \Leftrightarrow ⓐ \models \exists \vec{R}\phi_\alpha$$

(for simplicity, we neglect finite a). Moreover, ϕ_α is clearly a set-primitive recursive function of α and b, the parameter in the defining formula $\phi(x)$.

Next we form the Vaught approximations $\delta_{\alpha,\beta}$ to each $\exists \vec{R}\phi_\alpha$, for all $\beta < \omega_1$. $\delta_{\alpha,\beta}$ is a set-primitive recursive function of b, α, and β. On the basis of Vaught's theorem and the fact that $X \subset \mathscr{L}_{\omega_1}[b]$ we now have:

$$a \in X \Leftrightarrow (\mathscr{L}_{\omega_1}[b], \in \restriction \mathscr{L}_{\omega_1}[b]) \models$$

$$\exists A\,[A \text{ is admissible } \&\ b \in A\ \&\ a \in A$$

$$\&\ (\exists \alpha < o(A))[(\forall \beta < o(A)) ⓐ \models \delta_{\alpha,\beta}]]$$

which shows the second assertion of the theorem. Q.E.D.

Remark. Instead of applying 1.5, one can directly apply Corollary 1.2 to prove 1.6, in which case one gives a similar but cruder inductive argument than the one for 1.5.

2. Preliminaries on canonical Scott sentences and their approximations

In the next section, we will give some applications of Section 1 to canonical Scott sentences. Here we review definitions and some elementary properties. 2.15 and the explicit definition and use of certain formal notions are perhaps new.

Before we describe the actual construction of CSS($\mathcal{M}$), the canonical Scott sentence of $\mathcal{M}$, we note that for every structure $\mathcal{M}$, CSS($\mathcal{M}$) is a sentence of $L_{\infty\omega}$ (L is the language of $\mathcal{M}$),

$$N \models \mathrm{CSS}(\mathrm{M}) \quad \text{iff} \quad N \equiv_{\infty\omega} \mathcal{M}$$

and also, $\mathrm{CSS}(N) = \mathrm{CSS}(\mathcal{M})$ iff $N \equiv_{\infty\omega} \mathcal{M}$. References [21] and [22] contain a detailed study of canonical Scott sentences. The name "canonical Scott sentence" also comes from [21]. The construction itself is due to Chang [4]; it is also given in [10]. A fundamental result on canonical Scott sentences is Mark Nadel's theorem [22], stated as Corollary 3.14 below of which we will give two new proofs.

Fix a structure $\mathcal{M}$ of the language L. For a sequence $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ of elements of $|\mathcal{M}|$ (which may be the empty sequence) and for an arbitrary ordinal α, we define the formula

$$\phi_{\boldsymbol{a}}^{\alpha}(x_1, \ldots, x_n)$$

by induction on α as follows.

$$\phi^0_{\mathbf{a}} =_{\mathrm{df}} \bigwedge \mathrm{diag}_{\mathbf{a}}(x_1, \ldots, x_n)$$

where $\mathrm{diag}_{\mathbf{a}}$ is defined as the set

$$\{\theta(x_1, \ldots, x_n) \colon \mathcal{M} \models \theta(a_1, \ldots, a_n),\ \theta \text{ is an atomic or negated atomic formula of L}\}.$$

$$\phi^{\alpha+1}_{\mathbf{a}} =_{\mathrm{df}} \bigwedge_{a \in |\mathcal{M}|} \exists x_{n+1} \phi^{\alpha}_{\mathbf{a},a}(x_1, \ldots, x_n, x_{n+1}) \wedge \forall x_{n+1} \bigvee_{a \in |\mathcal{M}|} \phi^{\alpha}_{\mathbf{a},a}(x_1, \ldots, x_n, x_{n+1}),$$

and for λ a limit ordinal,

$$\phi^{\lambda}_{\mathbf{a}} =_{\mathrm{df}} \bigwedge_{\alpha<\lambda} \phi^{\alpha}_{\mathbf{a}}.$$

We will call $\phi^{\alpha}_{\mathbf{a}}$ the *canonical* α-*type of* $\mathbf{a}$ (*in* $\mathcal{M}$). For fixed α and $\mathcal{M}$, the $\phi^{\alpha}_{\mathbf{a}}$ collectively for all the $\mathbf{a}$ in $|\mathcal{M}|$ are called the canonical α-types *in* $\mathcal{M}$, and with a fixed length n of $\mathbf{a}$, the canonical α, n-types in $\mathcal{M}$. If we want to indicate $\mathcal{M}$ in the notation of canonical types, we write $(\phi^{\alpha}_{\mathbf{a}})^{\mathcal{M}}$.

The following lemma is due to Chang [4], cf. also the proof of Theorem 6 in [1].

Lemma 2.1. *The canonical* α-*types have quantifier rank* α *and they are complete with respect to* $L_{\infty\omega}$ *formulas of quantifier rank* $\leqslant \alpha$, *i.e.*

$$N \models \phi^{\alpha}_{\mathbf{a}}[b_1, \ldots, b_n] \Leftrightarrow (N, b_1, \ldots, b_n) \overset{\alpha}{\equiv}_{\infty,\omega} (\mathcal{M}, a_1, \ldots, a_n).$$

(Here $\overset{\alpha}{\equiv}_{\infty,\omega}$ signifies elementary equivalence with respect to $L_{\infty\omega}$ formulas of quantifier rank $\leqslant \alpha$).

Consider the smallest ordinal α such that

$$\phi^{\alpha}_{0} \wedge \bigwedge_{n<\omega} \bigwedge_{a_1,\ldots,a_n \in |\mathcal{M}|} \forall x_1 \cdots x_n [\phi^{\alpha}_{a_1,\ldots,a_n} \rightarrow \phi^{\alpha+1}_{a_1,\ldots,a_n}]$$

is true in $\mathcal{M}$. (For cardinality reasons, such an α must exist.) The displayed sentence with this smallest α is what is called the canonical Scott sentence of $\mathcal{M}$; it is denoted by $\mathrm{CSS}(\mathcal{M})$. α is called the *Scott height* of $\mathcal{M}$.

If $\mathcal{M}$ is a countable structure, then $\mathrm{CSS}(\mathcal{M})$ is an $L_{\omega_1\omega}$ sentence (uncountable structures and formulas will occur in the paper only passingly).

The notion to be introduced next, that of *formal canonical types* (f.c. types) (α-types, α, n-types), is intended to capture as much as possible of the notion proper purely syntactically. This notion is relative to a fixed language L.

Definition 2.2. The notion "$\phi(x_1, \ldots, x_n)$ is a f.c. α, n-type" is defined by induction as follows.

$\phi(x_1, \ldots, x_n)$, is a f.c. 0, n-type if it is a conjunction $\bigwedge \Theta$ such that each element of Θ is an atomic or negated atomic formula of L with the (free) variables $x_1, \ldots, x_n$ at most and such that for every atomic formula of L with the $x_1, \ldots, x_n$ at most, exactly one of θ and $\neg\theta$ belongs to Θ. (This means that Θ is a "diagram" "of $x_1, \ldots, x_n$".)

$\phi(\boldsymbol{x})$ ($\boldsymbol{x} = x_1, \ldots, x_n$) is a f.c. $\alpha + 1$, n-type iff for some set Φ of f.c. α, $n+1$-types, ϕ is the formula ϕ_Φ where ϕ_Φ is defined as

$$\bigwedge\{\exists x_{n+1}\phi : \phi \in \Phi\} \wedge \forall x_{n+1} \bigvee \Phi.$$

$\phi(\boldsymbol{x})$ is a f.c. λ, n-type, for λ a limit ordinal, if ϕ is

$$\bigwedge\{\phi_\alpha : \alpha < \lambda\}$$

for some $\phi_\alpha (\alpha < \lambda)$ such that each ϕ_α is a f.c. α, n-type.

Notice that each f.c. α, n-type is a formula of $L_{\infty\omega}$, it has precisely the free variables $x_1, \ldots, x_n$, and it has quantifier rank α. Also, it is clear that every canonical α, n-type is a f.c. α, n-type.

We denote by FC, FC_α, $FC_{\alpha,n}$ the class of all f.c. types, f.c. α-types, f.c. α, n-types respectively. An upper index A like in FC^A designates that we consider $FC \cap A$, e.t.c.

The next Lemma shows that the canonical types are exactly the consistent f.c. types.

Lemma 2.3. *Suppose that ϕ is a f.c. α, n-type and $\mathcal{M} \models \phi[a_1, \ldots, a_n]$. Then ϕ is identical to $\phi^\alpha_{\boldsymbol{a}}$, the canonical type of $\langle a_1, \ldots, a_n\rangle$ in $\mathcal{M}$.*

The proof is an easy induction. The cases $\alpha = 0$ and $\alpha = \lambda$ limit are trivial. Suppose $\Phi \subset FC_{\alpha,n+1}$ and $\mathcal{M} \models \phi_\Phi[\boldsymbol{a}]$. Then by the first conjunct in ϕ_Φ, for every $\phi(\boldsymbol{x}, x_{n+1})$ in Φ there is $a_{n+1} \in |\mathcal{M}|$ such that $\mathcal{M} \models \phi[\boldsymbol{a}, a_{n+1}]$, hence by the induction hypothesis, every $\phi \in \Phi$ is identical to $\phi^\alpha_{\boldsymbol{a},a_{n+1}}$ for some $a_{n+1} \in |\mathcal{M}|$, i.e.

$$\Phi \subset \{\phi^\alpha_{\boldsymbol{a},a_{n+1}} : a_{n+1} \in |\mathcal{M}|\}.$$

The opposite inclusion is shown using the second conjunct in ϕ_Φ. The equality of the two sets means that $\phi_\Phi = \phi^{\alpha+1}_{\boldsymbol{a}}$, q.e.d.

On the basis of the inductive definition of f.c. types, the next lemma is obvious.

Lemma 2.4. FC *is set-primitive recursive, hence* FC^A *is Δ-definable on an admissible A, in fact, uniformly. Also, for $\phi \in$ FC the unique α and n such that $\phi \in FC_{\alpha,n}$ are set-primitive recursive functions of ϕ.*

In order to be able to relate two notions of scatterdness below, we introduce a "logic-free" rudimentary version of canonical types. Let us define $\xi^\alpha_{\boldsymbol{a}}$ (or more precisely, $(\xi^\alpha_{\boldsymbol{a}})^{\mathcal{M}}$) for $\boldsymbol{a} = \langle a_1, \ldots, a_n\rangle$, $a_i \in |\mathcal{M}|$, $\mathcal{M}$ a fixed structure, as follows:

$$\xi^0_{\boldsymbol{a}} =_{\mathrm{df}} \mathrm{diag}^{\mathcal{M}}_{\boldsymbol{a}}(x_1, \ldots, x_n),$$

$$\xi^{\alpha+1}_{\boldsymbol{a}} =_{\mathrm{df}} \{\xi^\alpha_{\boldsymbol{a},a_{n+1}} : a_{n+1} \in |\mathcal{M}|\},$$

$$\xi^\lambda_{\boldsymbol{a}} =_{\mathrm{df}} \{\xi^\alpha_{\boldsymbol{a}} : \alpha < \lambda\} \text{ for limit } \lambda.$$

Put

$$\xi^{\alpha}(K) =_{\mathrm{df}} \{(\xi^{\alpha}_{\boldsymbol{a}})^{\mathcal{M}} : \mathcal{M} \in K,\ n \in \omega,\ \boldsymbol{a} \in {}^{n}|\mathcal{M}|\}$$

and

$$\xi(K) =_{\mathrm{df}} \bigcup_{\alpha \in \mathrm{ord}} \xi^{\alpha}(K)$$

for a class K of structures.

It is clear that $\xi^{\alpha}_{\boldsymbol{a}}$ contains all information $\phi^{\alpha}_{\boldsymbol{a}}$ does. In fact, we clearly have

Lemma 2.5. *There are set-primitive recursive functions f and f^{-1} such that $\phi^{\alpha}_{\boldsymbol{a}} = f(\xi^{\alpha}_{\boldsymbol{a}})$ and $\xi^{\alpha}_{\boldsymbol{a}} = f^{-1}(\phi^{\alpha}_{\boldsymbol{a}})$.*

Let K be any class of structures of the language L, let L_B be a countable fragment of $\mathrm{L}_{\omega_1\omega}$, and let n be an arbitrary natural number. By $S_n(K)$, more precisely, $S_n^{\mathrm{L}_B}(K)$, we denote the set of all complete n-types over L_B realized in some model in K. In other words, t is an element of $S_n(K)$ if t is a set of formulas in L_B with the free variables $x_1, \ldots, x_n$ at most and there is $\mathcal{M} \in K$ and $a_1, \ldots, a_n \in |\mathcal{M}|$ such that for $\phi(x_1, \ldots, x_n) \in \mathrm{L}_B$, $\phi(x_1, \ldots, x_n) \in t$ iff $\mathcal{M} \models \phi[a_1, \ldots, a_n]$.

Definition 2.6. Let A be any set, K any class of structures of L. K is called *A-scattered* if for all countable fragments $\mathrm{L}_B \in A$ and for all $n < \omega$, $S_n^{\mathrm{L}_B}(K)$ has power $< 2^{\aleph_0}$.

K is *scattered* (Morley [20]) if it is HC-scattered.

The following is quite elementary.

Lemma 2.7. *Let A be admissible. If K is A-scattered, then $\xi(K)$ is A-scattered in the sense of* 1.1.

Proof. We show that for any $b \in A$ and any α

$$\mathcal{P}(b) \cap \xi^{\alpha}(K)$$

has power $< 2^{\aleph_0}$. We leave the cases $\alpha = 0$ and α a limit ordinal to the reader and we handle the case $\alpha := \alpha + 1$. We use the primitive recursive set function f relating the $(\xi^{\alpha}_{\boldsymbol{a}})^{\mathcal{M}}$ and $(\phi^{\alpha}_{\boldsymbol{a}})^{\mathcal{M}}$, and we put $B =_{\mathrm{df}} f^*(b) \cap \mathrm{FC}^A_{\alpha}$. Clearly, $B \in A$.

For any

$$\xi \in \mathcal{P}(b) \cap \xi^{\alpha+1}(K),\ \xi = (\xi^{\alpha+1}_{\boldsymbol{a}})^{\mathcal{M}},$$

define

$$\bar{\xi} = \{\exists x_{n+1}\, \eta(\boldsymbol{x}, x_{n+1}) \colon \eta \in f^*(\xi)\}$$
$$\cup \{\neg \exists x_{n+1}\, \eta(\boldsymbol{x}, x_{n+1}) \colon \eta \in B - f^*(\xi)\}.$$

Since all elements of B are f.c. $\alpha, n+1$-types, by 2.3 we have that for $\eta \in B$, there is $a_{n+1} \in |\mathcal{M}|$ such that $\mathcal{M} = \eta[\boldsymbol{a}, a_{n+1}]$ iff $\eta \in f^*(\xi)$. This means that every element of $\bar{\xi}$ is true for $(\mathcal{M}, \boldsymbol{a})$, hence that $\bar{\xi}$ generates an element of $S_n^{L_{B'}}(K)$ for an appropriate fragment $L_{B'} \in A$. Since for different $\xi \in \xi^{\alpha+1}(K)$, the $\bar{\xi}$ are obviously contradictory, we have that $|\mathcal{P}(b) \cap \xi^{\alpha+1}(K)| \leq |S_n^{L_{B'}}(K)| < 2^{\aleph_0}$, showing our claim.

Next notice that

$$\mathrm{TC}(\xi(K)) = \bigcup_{\alpha \in \mathrm{ord}} \xi^{\alpha}(K) \cup \mathrm{TC}(\xi^0(K))$$

and also, for $b \in A$,

$$\mathcal{P}(b) \cap \mathrm{TC}(\xi(K)) = \Big(\mathcal{P}(b) \cap \bigcup_{\alpha < o(A)} \xi^{\alpha}(K)\Big)$$
$$\cup (\mathcal{P}(b) \cap \mathrm{TC}(\cup\, \xi^0(K))).$$

By what was shown above, and $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$, the first part of the union is of power $< 2^{\aleph_0}$. Clearly, $\mathrm{TC}(\cup\, \xi^0(K))$ is a subset of A. Thus indeed, $\mathcal{P}(b) \cap \mathrm{TC}(\xi(K))$ is of power $< 2^{\aleph_0}$. Q.E.D.

Remark 1. If K is a $\mathrm{PC}_{\omega_1\omega}$ class (and hence, has the downward Lowenheim–Skolem property) and has $< 2^{\aleph_0}$ isomorphism types of countable models, then both K and $\xi(K)$ are obviously scattered, so in this case 2.7 is of no importance. We happen to have an application, however, where we have to use the scatteredness notion of 2.6 in an essential way, viz. Theorem 4.7.

Next we introduce the notion of *formal canonical Scott sentence* (*f.c.S.s.*).

Definition 2.8. A sentence ϕ of $L_{\infty\omega}$ is a *f.c.S.s. of* L if it is of the form

$$\phi^0 \wedge \bigwedge_{n<\omega} \bigwedge_{i \in I_n} \forall x_1 \cdots x_n (\phi_i \to \psi_i)$$

and there is an ordinal α (called the *rank* of ϕ) such that

(i) for $i \in I_n$, ϕ_i is a f.c. α, n-type, $\psi_i =_{\mathrm{dt}} \phi_{\Phi_i}$ is a f.c. $\alpha+1, n$-type (for the notation ϕ_{Φ_i}, see the definition of f.c. types) and

(ii) for each n,

$$\bigcup_{i \in I_n} \Phi_i = \{\phi_i : i \in I_{n+1}\}$$

and

$$\{\phi^0\} = \{\phi_i : i \in I_0\}.$$

Let us denote by $\mathrm{CSS}_\alpha(\mathcal{M})$ the sentence displayed at the definition of $\mathrm{CSS}(\mathcal{M})$ but for an arbitrary ordinal α. (Thus $\mathrm{CSS}(\mathcal{M})$ is $\mathrm{CSS}_\alpha(\mathcal{M})$ for the smallest α such that $\mathcal{M} \models \mathrm{CSS}_\alpha(\mathcal{M})$.) Then clearly, $\mathrm{CSS}_\alpha(\mathcal{M})$ is a f.c.S.s. Notice that by Chang's proof of the Scott isomorphism theorem we have

Lemma 2.9. *For any $\mathcal{M}$ and any ordinal α, $N \models \mathrm{CSS}_\alpha(\mathcal{M})$ implies $N \equiv_{\infty\omega} \mathcal{M}$.*

We have the following analogue of 2.3:

Lemma 2.10. *Suppose that ϕ is a* f.c.S.s. *with rank α and that $\mathcal{M} \models \phi$. Then*

$$\phi = \mathrm{CSS}_\alpha(\mathcal{M}).$$

Proof. By 2.3, we have $\phi^0 = \phi_0^\alpha = \phi_i (i \in I_0)$. Also, for every $i \in I_0$, $\mathcal{M} \models \psi_i$, hence again by 2.3, $\psi_i = \phi_0^{\alpha+1}$ for each $i \in I_0$. By induction on n, we show that, in general,

$$\{\langle \phi_i, \psi_i \rangle : i \in I_n\} = \{\langle \phi_a^\alpha, \phi_a^{\alpha+1} \rangle : a \in^n |\mathcal{M}|\}.$$

Suppose that this is true for n. It follows that

$$\bigcup_{i \in I_n} \Phi_i = \{\phi^\alpha_{a_1,\dots,a_n,a_{n+1}} : a_1, \dots, a_{n+1} \in |\mathcal{M}|\},$$

hence by the first equality in 2.8 (ii)

$$\{\phi_i : i \in I_{n+1}\} = \{\phi^\alpha_{a_1,\dots,a_{n+1}} : a_1, \dots, a_{n+1} \in |\mathcal{M}|\}.$$

Pick an $i \in I_{n+1}$; $\phi_i = \phi_a^\alpha$ for some $a \in^{n+1} |\mathcal{M}|$. Since clearly $\mathcal{M} \models \phi_a^\alpha[a]$, by $\mathcal{M} \models \phi$ we have that $\mathcal{M} \models \psi_i[a]$. By 2.3 and 2.8, this implies that $\psi_i = \phi_a^{\alpha+1}$. This, together with the last displayed equality, shows the claimed equality for $n + 1$.

This completes the proof of 2.10.

The following is obvious.

Lemma 2.11. *The predicate "($\phi \in A$ and) ϕ is a formal canonical Scott sentence" is Δ on any admissible A (in fact, uniformly). Also, the rank of a f.c.S.s. ϕ is a set-primitive recursive function of ϕ.*

For the purposes of 2.15 below, we introduce one more simple tool.

Definition 2.12. Given the language L, we define by induction on the ordinal α the formulas $\theta_n^\alpha(x_1, \dots, x_n; y_1, \dots, y_n)$ as follows

$$\theta_n^0(x_1, \dots, x_n; y_1, \dots, y_n) = \bigwedge\{\theta(x_1, \dots, x_n) \Leftrightarrow \theta(y_1, \dots, y_n):$$

$$\theta \text{ is an atomic formula of L with the (free) variables}$$

$$x_1, \dots, x_n \text{ at most}\},$$

$$\theta_n^{\alpha+1}(\boldsymbol{x}, \boldsymbol{y}) = \forall x_{n+1} \exists y_{n+1}\, \theta_{n+1}^\alpha(\boldsymbol{x}, x_{n+1}; \boldsymbol{y}, y_{n+1})$$

$$\wedge \forall y_{n+1} \exists x_{n+1}\, \theta_{n+1}^\alpha(\boldsymbol{x}, x_{n+1}; \boldsymbol{y}, y_{n+1}),$$

$$\theta_n^\lambda(\boldsymbol{x}, \boldsymbol{y}) = \bigwedge_{\alpha<\lambda} \theta_n^\alpha(\boldsymbol{x}, \boldsymbol{y}) \text{ for limit } \lambda.$$

The following have trivial proofs.

Lemma 2.13. *For a structure $\mathcal{M}$ of* L, *for any $n < \omega$, any ordinal α and $\boldsymbol{a}, \boldsymbol{b} \in^n |\mathcal{M}|$, we have*

$$(\mathcal{M}, \boldsymbol{a}) \overset{\alpha}{\equiv}_{\infty,\omega} (\mathcal{M}, \boldsymbol{b}) \Leftrightarrow \mathcal{M} \models (\phi^{\alpha}_{\boldsymbol{a}})^{\alpha}[\boldsymbol{b}]$$

$$\Leftrightarrow \mathcal{M} \models \theta^{\alpha}_{n}[\boldsymbol{a}\,;\boldsymbol{b}]$$

(see 2.1 for the first equivalence).

Lemma 2.14. *ϑ^{α}_{n} is a set-primitive recursive function of* L, *α and n.*

We are now in the position of being able to prove

Theorem 2.15. *The predicate "($\phi \in A$ and) ϕ is the canonical Scott sentence of some structure" is Δ on any admissible $A \subset$ HC such that $\omega \in A$, in fact, uniformly.*

Proof. Consider the the following sentence:

$$\sigma_{\alpha} = \bigwedge_{n<\omega} \forall x_1 \cdots x_n\, y_1 \cdots y_n [\theta^{\alpha}_{n} \rightarrow \theta^{\alpha+1}_{n}].$$

By 2.13 and the definition of $\mathrm{CSS}_{\alpha}(\mathcal{M})$, we have that for any $\mathcal{M}$ of L,

$$\mathcal{M} \models \mathrm{CSS}_{\alpha}(\mathcal{M}) \Leftrightarrow \mathcal{M} \models \sigma_{\alpha}.$$

Hence by 2.10 and the definition of CSS($\mathcal{M}$) we have that for any $\mathcal{M}$ of L,

$\phi = \mathrm{CSS}(\mathcal{M}) \Leftrightarrow \phi$ is a f.c.S.s. with rank α and

$$\mathcal{M} \models \phi \wedge \bigwedge_{\beta<\alpha} \neg\, \sigma_{\beta}.$$

Denote the set of $\phi \in \mathrm{L}_A$ such that ϕ is CSS(M) for some $\mathcal{M}$ by CSS^A. We have

$\phi \in \mathrm{CSS}^A \Leftrightarrow \phi \in \mathrm{L}_A$ & ϕ is a f.c.S.s.

& $\phi \wedge \bigwedge_{\beta<\alpha} \neg\, \sigma_{\beta}$ is consistent

where α is the rank of ϕ.

Assume A is admissible, $A \subset$ HC, and $\omega \in A$. Then by 2.14, σ_{β} is an A-recursive function of β. Hence by 2.11 and the Barwise completeness theorem, CSS^A is Π on A. It remains to show that CSS^A is also Σ.

Let ϕ be an arbitrary f.c.S.s. in L_A and let ϕ' be $\phi \wedge \bigwedge_{\beta<\alpha} \neg\, \sigma_{\beta}$, α = rank of ϕ. We claim that

ϕ' is consistent $\Leftrightarrow$ there is $S \in A$ such that

S is a consistency property and

for some $s \in S$, $\phi' \in s$.

From right to left, this is a consequence of the model existence theorem. Conversely, if ϕ' is consistent, then by the above, ϕ is CSS($\mathcal{M}$) for some $\mathcal{M}$, hence ϕ and a fortiori ϕ' are complete with respect to $L_{\infty\omega}$. It follows that the canonical consistency property associated with ϕ', $S(L_B, \phi')$ in the notation of Lemma 1 in [23] where $L_B \in A$ is the smallest fragment containing ϕ', is actually an element of A. This establishes the claim.

Since the notion of a consistency property is Δ_0, the last equivalence, together with the above characterization of CSS^A and 2.11, gives that CSS^A is indeed Σ on A.

This completes the proof of 2.15.

Remark 2. For the case A is locally countable, there is another proof of 2.15 which, though it is not really simpler than the one given (how could it be?), follows perhaps more familiar lines. We sketch this proof as follows. Firstly, there is a set theoretical formula,

"ϕ is CSS($\mathcal{M}$)"

in two variables ϕ and $\mathcal{M}$ expressing the intended meaning (i.e., $(V, \in) \models$ "ϕ is CSS($\mathcal{M}$)" $\Leftrightarrow$ $\mathcal{M}$ is a structure and ϕ is CSS($\mathcal{M}$)) such that "ϕ is CSS($\mathcal{M}$)" is absolute with respect to any two models (well-founded or not) $\mathcal{N}_1$ and $\mathcal{N}_2$ of KP + Axiom of Infinity with $\mathcal{N}_2$ end extending $\mathcal{N}_1$. Consider the following A-finite theory T_ϕ associated recursively with the sentence ϕ in A ($\boldsymbol{M}$ is a new constant, $L(\phi)$ is the set of non-logical symbols in ϕ):

$$\left.\begin{array}{l} \mathrm{KP} \\ \boldsymbol{a} \neq \boldsymbol{b} \\ \forall x\,[x \in \boldsymbol{a} \leftrightarrow \bigvee\{x = \boldsymbol{c} : c \in a\}] \end{array}\right\} a \neq b,\ a, b \in \{\phi, L(\phi)\} \cup \mathrm{TC}(\phi)$$

"$\boldsymbol{M}$ is a structure of $L(\phi)$, $|\boldsymbol{M}| = \omega$",

"$\boldsymbol{\phi}$ is CSS($\boldsymbol{M}$)".

Then we have that for $\phi \in A$,

$$\phi \in \mathrm{CSS}^A \Leftrightarrow T_\phi \text{ is consistent.}$$

It follows that CSS^A is Π on A. On the other hand, in case A is locally countable,

$$\phi \in \mathrm{CSS}^A \Leftrightarrow (A, \in \restriction A) \models \exists \mathcal{M} \text{ "} \phi \text{ is CSS}(\mathcal{M})\text{"}.$$

From left to right, one uses, as in the above proof, that the canonical consistency property associated with ϕ is an element of A and that the model existence theorem is true within the locally countable A. Thus, if $\phi = \mathrm{CSS}(\mathcal{M}')$ and $\phi \in A$, then there is $\mathcal{M} \in A$ such that $\mathcal{M} \models \phi$; but then $\mathcal{M} \equiv_{\infty\omega} \mathcal{M}'$ and hence $\phi = \mathrm{CSS}(\mathcal{M})$.

Can this last proof be lifted to the general case? Even if the answer is "yes", the "formal" notions introduced above seem to be of some interest.

3. Canonical Scott sentences of models in scattered classes

As the first application, we will give a proof of a theorem of G. Sacks, Theorem 3.3 (private communication without proof, it will appear in [25]). Although the theorem does not mention Scott sentences, our proof will use them.

Definition 3.1. (G. Sacks). Let A be an admissible set, $\mathcal{M}$ a structure. We say that $\mathcal{M}$ *has enough A-finite automorphisms* if for any two finite sequences $\boldsymbol{a} = \langle a_1, \dots, a_n\rangle$ and $\boldsymbol{b} = \langle b_1, \dots, b_n\rangle$ of elements of $|\mathcal{M}|$, if $\mathcal{M}$ has an automorphism carrying $\boldsymbol{a}$ into $\boldsymbol{b}$, then $\mathcal{M}$ has an A-finite automorphism carrying $\boldsymbol{a}$ into $\boldsymbol{b}$.

Example 3.2. Suppose that $\mathcal{M} \in A \subseteq \mathrm{HC}$, and $\mathcal{M}$ has only countably many automorphisms. Then, by an easy application of 1.2. every automorphism of $\mathcal{M}$ is A-finite, hence a fortiori $\mathcal{M}$ has enough A-finite automorphisms.

Also, obviously every rigid structure $\mathcal{M}$ in A satisfies 3.1.

Assume throughout this section that A is a countable admissible set, $\omega \in A$, L' is a language Δ-definable on A, $\mathrm{L} \subset \mathrm{L}'$ and L is A-finite.

Theorem 3.3 (G. Sacks). *Suppose that A is locally countable. Suppose that Σ is a Σ-definable theory in L'_A. Let K be defined by the Σ^1_1 sentence $\exists(\mathrm{L}' - \mathrm{L}) \wedge \Sigma$. Suppose that*

(i) *K has $< 2^{\aleph_0}$ isomorphism types of countable members, and*

(ii) *each $\mathcal{M} \in K$ with $|\mathcal{M}| = \omega$ has enough B-finite automorphisms where B is the smallest admissible set such that $\mathcal{M} \in B$ and $A \subset B$.*

Then every countable model in K has an isomorphic copy in A.

Before turning to the proof, we mention an equivalent way of expressing Definition 3.1, due to Nadel [22], Theorem 3.4 (cf. also [21]).

Lemma 3.4. *For $\mathcal{M} \in A$ and $|\mathcal{M}|$ countable in the sense of A, $\mathcal{M}$ has enough A-finite automorphisms if and only if $\mathrm{CSS}(\mathcal{M}) \in A$ (the 'only if' direction does not need the assumption that $|\mathcal{M}|$ is countable within A).*

Example 3.5. Together with another result of Nadel's [22], Theorem 4.9 (cf. also [21]), this gives further examples satisfying 3.1: every scattered linear ordering (one that does not have a dense subordering) in A has its CSS in A.

Proof of 3.3. We will prove that under the hypotheses $(*)$ $\mathrm{CSS}(K) =_{\mathrm{df}} \{\mathrm{CSS}(\mathcal{M}) : \mathcal{M} \in K\}$ is a subset of an A-finite set. This will not require the hypothesis that A is locally countable. With this additional hypothesis, the conclusion follows from $(*)$ by the argument used at the end of Remark 2 in Section 2. The proof of $(*)$ consists in a direct application of the main Lemma 1.4. Consider the following theory T' in the language of set theory augmented with constants $\boldsymbol{a}$

for elements a of A, two new constants $\boldsymbol{M}$ and c, and further new constants $\boldsymbol{R}$ for $R \in \mathrm{L}' - \mathrm{L}$:

ZF,

diag(A),

"$\boldsymbol{M}$ is an L-structure with $|\boldsymbol{M}| = \omega$",

"$\boldsymbol{M}$, augmented with the new relations $\boldsymbol{R}$ such that $\boldsymbol{R}$ occurs in ϕ, satisfies ϕ" (for every $\phi \in \mathrm{T}'$),

"c is CSS($\boldsymbol{M}$)".

Here "c is CSS($\boldsymbol{M}$)" is a formula with properties described in Remark 2 in Section 2. It can be taken, for example, to be a formalization of

$$\text{“}c \text{ is a f.c. S.s. with rank } \alpha \text{ and } \mathcal{M} \models c \wedge \bigwedge_{\beta<\alpha} \neg \sigma_\beta\text{”},$$

cf. Section 2.

Clearly, T′ satisfies the (meager) assumptions made prior to 1.4. Let $\mathcal{N}$ be a normal model of T′. Let $\mathcal{M} = \boldsymbol{M}^{\mathcal{N}}$. Since $A' =_{\mathrm{df}} \mathrm{WF}(\mathcal{N})$ is an admissible set containing $\mathcal{M}$ and extending A (cf. e.g. [2]), by assumption (ii) of the theorem together with 3.4 we have that $\mathrm{CSS}(\mathcal{M}) \in A'$. Since ZF$\models$ "there is at most one CSS of any structure", using also the properties of the formula "c is CSS($\boldsymbol{M}$)" we infer that $c^{\mathcal{N}} = \mathrm{CSS}(\mathcal{M})$. This shows that (i) in 1.4 is satisfied and also that $\mathrm{tr}(\mathrm{T}') \subset \mathrm{CSS}(K) =_{\mathrm{df}} \{\mathrm{CSS}(\mathcal{M}) : \mathcal{M} \in K\}$. The converse of the last conclusion is easily seen. Hence $\mathrm{CSS}(K) = \mathrm{tr}(\mathrm{T})$, thus by assumption (i), tr(T) is scattered. The conclusion of 1.4 is that CSS(K) is a subset of an A-finite set, Q.E.D.

Remark 1. Notice that Theorem 3.3 is a generalization of the "admissible" version of the well-known theorem on non-characterisability of well-order. Notice also that by the first part of the proof of 1.4 giving that the ranks of members of tr(T) are bounded by some ordinal in A, if we delete hypothesis (i) from 3.3, we can still infer that the Scott height of $\mathcal{M} \in K$ (the ordinal α such that $\mathrm{CSS}(\mathcal{M}) = \mathrm{CSS}_\alpha(\mathcal{M})$) is bounded by some ordinal in A. An essentially equivalent result was obtained independently of G. Sacks and by a different method in [7]. Notice that by Example 3.5 this result generalizes the version of non-characterisability of well-order which mentions a copy of the rationals (Theorem 12 in [10]).

Remark 2. If we replace hypothesis (ii) by "every countable $\mathcal{M} \in K$ has only countably many automorphisms", then we can delete the hypothesis that A is locally countable. Namely, first of all, if the countable model $\mathcal{M}$ is rigid, then $\mathcal{M}$ has an isomorphic copy which is in the smallest admissible set A containing CSS($\mathcal{M}$), regardless if CSS($\mathcal{M}$) is countable in the sense of A. This holds essentially because the canonical α-types (with α the Scott height of $\mathcal{M}$) of distinct elements of $\mathcal{M}$ are distinct in this case and thus the universe of the copy of $\mathcal{M}$ can be taken as the set of canonical α, 1-types appearing in CSS($\mathcal{M}$). The relations between the elements of this universe can be easily recovered so that $a \to (\phi_a^\alpha)^{\mathcal{M}}$ will become an isomorphism of $\mathcal{M}$ onto some $\mathcal{M}' \in A$. Once we know this, then we have the same conclusion under the weaker condition that A has only countably many automorphisms, by using a result of Kueker's [11]. Hence the statement (*) in the proof of 3.3 implies what we want.

In the next Theorem 3.8 we want to draw weaker but related conclusions from weaker hypotheses.

Recall the notion of an A-scattered (scattered) class K of models, Definition 2.6. We will state a theorem, Theorem 3.20 below, which is a refinement of M. Morley's following theorem:

Theorem 3.6. ([20]). *If K is a scattered* $\mathrm{PC}_{\omega_1\omega}$ *class (one defined by a Σ^1_1 sentence over* $\mathrm{L}_{\omega_1\omega}$*) (in particular, if K has $< 2^{\aleph_0}$ non-isomorphic countable models), then K has at most $\aleph_1$ non-isomorphic countable members.*

Before that we will work with the weaker hypothesis of A-scatteredness for a fixed A.

Notation 3.7. For a structure $\mathcal{M}$, an ordinal α and a natural number n, $\rho_{\alpha,n}(\mathcal{M})$ will denote the set of all canonical α, n-types realized in $\mathcal{M}$,

$$\rho_{\alpha,n}(\mathcal{M}) = \{(\varphi^{\alpha}_{a})^{\mathcal{M}} : a \in^{n} |\mathcal{M}|\}.$$

Also,

$$\rho_{\alpha}(\mathcal{M}) = \bigcup_{n<\omega} \rho_{\alpha,n}(\mathcal{M}),$$

$$\rho_{\leq\alpha}(\mathcal{M}) = \bigcup_{\beta\leq\alpha} \rho_{\beta}(\mathcal{M}),$$

$$\rho_{<\alpha}(\mathcal{M}) = \bigcup_{\beta<\alpha} \rho_{\beta}(\mathcal{M}),$$

$$\rho_{\alpha}(K) = \bigcup_{\mathcal{M}\in K,\, n<\omega} \rho_{\alpha,n}(\mathcal{M}),$$

$$\rho_{<\alpha}(K) = \bigcup_{\beta<\alpha} \rho_{\beta}(K).$$

Also, recall the "ξ-notation" introduced before 2.5 and the definition of the set-primitive recursive function f in 2.5.

Theorem 3.8. *Recall that* $\mathrm{L} \in A$. *Assume that K is the class defined by* $\exists(\mathrm{L}' - \mathrm{L}) \wedge \Sigma$, *with Σ a Σ-definable theory in* L'_A. *Suppose that K is A-scattered. Then for each $\alpha < o(A)$, $\rho_{<\alpha}(K)$ is a subset of an A-finite set. In particular, the canonical α-types realized in models in K are all A-finite, for all $\alpha < o(A)$.*

Proof. We will represent $\xi_{<\alpha}(K)$ as $\mathrm{tr}(\mathrm{T})$ for a suitable T. T is defined as the collection of the following axioms:

KP,
$\mathrm{diag}_{\infty}(A)$,
"$\boldsymbol{M}$ is an L-structure, $|\boldsymbol{M}| = \omega$",
"$\boldsymbol{a}$ is a finite sequence of elements of $|\boldsymbol{M}|$",
"$f(c)$ is $(\phi_a^\beta)^M$ for some $\beta < \alpha$",
"$\boldsymbol{M} \in K$"
(the last part is expressed more precisely in the proof of 3.3). Now we have $\mathrm{tr}(\mathrm{T}) = \xi_{<\alpha}(K)$ and that $r(c^{\mathcal{N}}) < \alpha$, hence $c^{\mathcal{N}} \in \mathrm{WF}(\mathcal{N})$, for $\mathcal{N} \models \mathrm{T}$. By 2.7, tr(T) is A-scattered. Hence by 1.4, $\xi_{<\alpha}(K)$ is a subset of an A-finite set. Applying the set-primitive recursive function f (hence $f \restriction A$ is A-recursive), $f^*(\xi_{<\alpha}(K)) = \rho_{<\alpha}(K)$ and we obtain the desired conclusion. Q.E.D.

We introduce some further notation.

Notation 3.9.

$$\tau_{<\alpha}(K) =_{\mathrm{df}} \{\rho_{<\alpha}(\mathcal{M}) : \mathcal{M} \in K\},$$

$$\mathrm{CSS}^{<\alpha}(K) =_{\mathrm{df}} \{\phi : \phi = \mathrm{CSS}(\mathcal{M}) = \mathrm{CSS}_\beta(\mathcal{M}) \text{ for some } \quad \mathcal{M} \in K,\ \beta < \alpha\}$$

i.e., $\mathrm{CSS}^{<\alpha}(K)$ is the set of canonical Scott sentences of models in K with Scott height $< \alpha$.

Theorem 3.10. *Under the hypotheses of 3.8, we have that both $\tau_{<\alpha}(K)$ and $\mathrm{CSS}^{<\alpha}(K)$ are subsets of A-finite sets, for all $\alpha < o(A)$.*

Proof. (a) Fix some $\alpha < o(A)$. Take some $a \in A$ such that $\rho_{<\alpha}(K) \subset a$ (by 3.8). By using 2.4 and Δ separation, also make sure that every element of a is a f.c. β-type for some $\beta < \alpha$. Then we can write

$$x \in \tau_{<\alpha}(K) \Leftrightarrow x \subset a \text{ and there is } \mathcal{M} \in K \text{ such that for all } \phi \in a,\ \phi$$
$$\text{is realized (i.e. } \exists x_1 \cdots x_n \phi \text{ is true if } \phi \text{ is an } n\text{-type) in } \mathcal{M} \text{ iff } \phi \in x.$$

This shows that $\tau_{<\alpha}(K)$ is a Σ_1^1 subset of a relative to A, in the terminology introduced before 1.2. Using the map $x \to \bar{x}$ defined by

$$\bar{x} = \{\exists x_1 \cdots x_n \phi : \phi \in x, \quad \phi \text{ is an } n\text{-type}, n < \omega\}$$
$$\cup \{\neg \exists x_1 \cdots x_n \phi : \phi \in a - x, \quad \phi \text{ is an } n\text{-type}, n < \omega\}$$

and using 2.3 we see that (i) $x \in \tau_{<\alpha}(K)$ implies that for some $\mathcal{M} \in K$, each element of $\bar{x}$ is true in $\mathcal{M}$ and (ii) for different $x_1, x_2 \subset a$, $\bar{x}_1$ and $\bar{x}_2$ are contradictory. Hence, from the A-scatteredness of K it follows that $\tau_{<\alpha}(K)$ has power $< 2^{\aleph_0}$. The conclusion for $\tau_{<\alpha}(K)$ follows from 1.2.

(b) The conclusion for $\mathrm{CSS}^{<\alpha}(K)$ follows from (a) after a little bit of nasty work.

What we show is that once we know that the Scott height of $\mathcal{M}$ is $< \alpha$, $\mathrm{CSS}(\mathcal{M})$ can be recursively obtained from $\rho_{<\alpha}(\mathcal{M})$. For every f.c. $\beta + 1$, n-type ψ there is at least one f.c. β, n-type ϕ such that $\models \psi \to \phi$; in case ψ is consistent, i.e. $\psi = (\phi_a^{\beta+1})^{\mathcal{M}}$ for some $\mathcal{M}$ and a, ϕ is unique and $\phi = (\phi_a^{\beta})^{\mathcal{M}}$ (see 2.3); in case ψ is inconsistent, every ϕ works. Hence, by using Σ-collection, we obtain that

$$(A, \in \restriction A) \models \forall a\, \exists b\, [\forall \beta\, \forall n\, \forall \psi\, [\text{``}\psi \text{ is a f.c. } \beta+1,\ n\text{-type''} \to$$
$$(\exists \phi \in b)(\exists d \in b)[\text{``}\phi \text{ is a f.c. } \beta,\ n\text{-type and}$$
$$d \text{ is a derivation of } \psi \to \phi\text{''}]].$$

For fixed a, and some b obtained from the last fact, consider the function $g \in A$ such that $\operatorname{dom} g = a$ and for $\psi \in a$,

$$g(\psi) = \phi \quad \text{if } \psi \text{ is a f.c. } \beta+1,\ n\text{-type for some } \beta \text{ and } n \text{ and}$$
$$\phi \text{ is the } \textit{unique} \text{ f.c. } \beta,\ n\text{-type such that}$$
$$\exists d \in b[d \text{ is a derivation of } \psi \to \phi],$$
$$g(\psi) = 0 \quad \text{otherwise.}$$

Now let α be a limit ordinal and let a be an A-finite set including $\rho_{<\alpha}(K)$, $\tau_{<\alpha}(K)$ as subsets. Let b and g be defined as above for a. Clearly, every element x of $\tau_{<\alpha}(K)$ is a subset of $\rho_{<\alpha}(K)$, hence a subset of a too. Now, let c be the set of all syntactical objects of the form

$$\phi_0 \wedge \bigwedge\{(\forall)[g(\psi) \to \psi] \colon \psi \in x \cap a \cap \mathrm{FC}_{\beta+1}\}$$

for some $x \in a$, $\varphi_0 \in x$ and $\beta < \alpha$; here $\mathrm{FC}_{\beta+1}$ denotes the class of f.c. $\beta+1$-types and $(\forall)$ signifies universal closure. Inspection of our procedure reveals that $\mathrm{CSS}^{<\alpha}(K) \subset c$. Also, clearly $c \in A$.

Remark 3. Under the stronger hypothesis that K has $< 2^{\aleph_0}$ non-isomorphic countable members, 3.10 has a, by now, straightforward proof using 1.4.

Remark 4. In Sacks' theorem 3.3, we can replace condition (i) by the apparently weaker one that K is A-scattered. This is a consequence of 3.10 and of the fact that using (ii) alone, we obtain that $\mathrm{CSS}(K) = \mathrm{CSS}^{<\alpha}(K)$ for some $\alpha \in A$ (see Remark 1).

To draw conclusions from 3.10, we state two Lemmas not mentioning admissible sets. Recall the notion of prime model from [10].

Lemma 3.11. *Let the theory $\mathrm{T} \subset \mathrm{L}_B$ be scattered with respect to the fragment L_B, i.e., $S_n^{\mathrm{L}_B}(\mathrm{Mod}\,\mathrm{T})$ countable for each $n < \omega$. Then each L_B-complete extension of T has a prime model.*

Proof. (Cf. Corollary C, p. 64 in [10].)

The following simple but important observation was made by Nadel in [22] in the course of the proof of his theorem stated as Corollary 3.14 below.

Lemma 3.12. ([22]). *If $\mathcal{M}$ is the prime model of some complete theory in the countable fragment* L_B, *then $\mathcal{M}$ has Scott height* $\leq \sup\{\text{quantifier rank }(\psi)\colon \psi \in L_B\}$.

The proof uses 2.1.

By combining 3.11, 3.12 and 3.10, we immediately obtain

Corollary 3.13. *For an A-finite fragment L_B and a theory T in L_B, Σ-definable on A, if $K =_{df} \mathrm{Mod}\,T$ is A-scattered and non-empty, then it has a member with canonical Scott sentence in A. Moreover, the rank of this canonical Scott sentence is bounded as in* 3.12.

In turn, 3.13 immediately implies

Corollary 3.14. (Nadel's Theorem 5.1, [22]). *Suppose that the sentence $\phi \in A$ is $\aleph_0$-categorical. Then the canonical Scott sentence of the (countable) model of ϕ is an element of A and has rank $\leq$ (the quantifier rank of ϕ) $+\,\omega$.*

Conversely, using the methods of [23], 3.13 can be derived from 3.14, in fact, with the slightly weaker hypothesis that T is scattered with respect to L_B.

Let us denote the set $\{\mathrm{CSS}(\mathcal{M}) \subset A\colon \mathcal{M}$ any structure$\}$ by CSS^A. The following was obtained in [23] as part of Theorem 6 for locally countable A. Using 3.14, it could be proved in full by the methods of [23].

Corollary 3.15. *A sentence $\phi \in L_A$ (or a Σ-definable theory ϕ in an A-finite fragment) has exactly n non-isomorphic countable models (n is a fixed natural number) if and only if $\models \phi \leftrightarrow \bigvee_{i<n} \phi_i$ for n distinct $\phi_i \in \mathrm{CSS}^A$. In fact, the ranks of the ϕ_i can be bounded by $\alpha + \omega^2$ where α is the quantifier rank of ϕ.*

The proof is an obvious induction based on 3.13.

Recall that CSS^A is Δ on A (2.15). Hence 3.15 can be considered as a "syntactic characterization result". In particular,

Corollary 3.16. *For a fixed n, the predicate for $\phi \in L_A$: "ϕ has exactly n countable models" is Σ on A, in fact uniformly for all admissible $A \subseteq \mathrm{HC}$ with $\omega \in A$.*

Similarly, "ϕ has at most n countable models", "ϕ has finitely many countable models" are Σ on A.

Using 2.15 we can strengthen 3.10 for the special case $K = \mathrm{Mod}(\phi)$ with a single sentence $\phi \in L_A$ as follows.

Corollary 3.17. *For $K = \mathrm{Mod}(\phi)$ with $\phi \in L_A$, if K is A-scattered, then $\mathrm{CSS}^{<\alpha}(K) \in A$ for any $\alpha < \mathrm{o}(A)$, and $\mathrm{CSS}^{<\mathrm{o}(A)}(K)$ is Δ on A.*

Also, $\rho_{<\alpha}(K) \in A$ and $\tau_{<\alpha}(K) \in A$ for $\alpha < \mathrm{o}(A)$.

Proof. $\sigma \in \mathrm{CSS}^{<o(A)}(K)$ iff $\sigma \in \mathrm{CSS}^A$ and $\sigma \vdash \phi$

iff $\sigma \in \mathrm{CSS}^A$ and $\sigma \vdash \neg\phi$,

which shows the second assertion. From this and 3.10 the first assertion follows. The proof of the rest uses 3.13.

Corollary 3.18. *For K as in 3.17, the L_A-theory of K is A-decidable* (Δ *on* A).

Proof. We have to show that for an L_A sentence to be consistent with ϕ is a condition that is Σ on A. But if ψ is consistent with ϕ, then $\phi \wedge \psi$ is a consistent A-scattered theory, hence by 3.13 it has a model with canonical Scott sentence in A. Hence ψ is consistent with ϕ iff $(\exists\sigma \in \mathrm{CSS}^A)(\sigma \vdash \psi \wedge \phi)$. By 2.15, the assertion follows.

Remark 5. Given an $\aleph_0$-categorical $\phi \in \mathrm{L}_A$, Nadel's proof of 3.14 gives a CSS in A equivalent to ϕ in a 'recursive way'. It follows from 3.14 and 2.15 that, in fact, there is a partial recursive function on A whose domain is precisely the set of $\aleph_0$-categorical sentences in A, and whose value at ϕ, if defined, is a CSS equivalent to ϕ.

Remark 6. There is another proof of 3.14, which is based on Theorem 1.3, Chapter IV in [2]. According to this theorem, if A is of the form $\mathscr{L}_\alpha[a]$, if T is a Σ-definable consistent theory in L_A with a language L containing $\in$ such that the axiom of extensionality is in T and if a fixed standard set x occurs as an element of the standard part $\mathrm{WF}(\mathcal{M})$ of *every* normal $\mathcal{M} \models \mathrm{T}$, then $x \in A$.

To apply this theorem for proving 3.18, we first note that we may assume that A is of the special form assumed here. Let $\phi \in A$ be $\aleph_0$-categorical. Consider T defined as follows:

KP,

$\mathrm{diag}(A)$,

"$\boldsymbol{M}$ is an L-structure, $|\boldsymbol{M}| \subset \omega$",

"$\boldsymbol{M} \models \phi$".

Let $\mathcal{M} \models \phi$. By 3.11 and 3.12, $\mathrm{CSS}(\mathcal{M})$ has rank $< o(A)$. It follows from this and the recursive nature of the definition of $\mathrm{CSS}(\mathcal{M})$ that for every model $\mathcal{N}$ of T, the construction of $\mathrm{CSS}(\boldsymbol{M}^{\mathcal{N}})$ takes place entirely in $\mathrm{WF}(\mathcal{N})$, i.e. $\mathrm{CSS}(\boldsymbol{M}^{\mathcal{N}}) \in \mathrm{WF}(\mathcal{N})$. But of course, $\mathrm{CSS}(\boldsymbol{M}^{\mathcal{N}}) = \mathrm{CSS}(\mathcal{M})_{\mathrm{df}} = x$ for the fixed $\mathcal{M}$. Now the assertion follows from Barwise's theorem.

Remark 7. We make some comments on Theorem 6 in [23]. Let L_B be an A-finite fragment and T a Σ-definable A-scattered theory in L_B. Nadel defines Γ_T as the set of canonical Scott sentences in A which have models in A itself. Then he shows that for (i) locally countable A and (ii) for A-finite T,

(*) $\Gamma_T \in A$ implies $\vdash T \leftrightarrow \bigvee \Gamma_T$,

(**) $\Gamma_T \notin A$ implies $\nvdash T \leftrightarrow \bigvee \Gamma_T$.

First of all, for locally countable A, Γ_T equals to $\mathrm{CSS}^A(T) =_{df} \{\mathrm{CSS}(\mathcal{M}): \mathcal{M} \models T, \mathrm{CSS}(\mathcal{M}) \in A\}$. (Of course, our main contribution in this respect is that $\mathrm{CSS}^A(T) = \mathrm{CSS}^{<o(A)}(T)$.) Now, assuming (ii) (but not necessarily (i)), by 3.13 we have:

(*)′ $\mathrm{CSS}^A(T) \in A$ implies $\vdash T \leftrightarrow \bigvee \mathrm{CSS}^A(T)$, generalizing (*). But more is true, namely even without (i) or (ii) or both, we have that

(*)″ $\mathrm{CSS}^A(T)$ is a subset of on A-finite set implies that $\vdash T \leftrightarrow \bigvee \mathrm{CSS}^A(T)$.
To see this, let the A-finite set in question be Φ. We can assume (by 2.15) that $\Phi \subset \mathrm{CSS}^A$. Let $(\neg)\Phi$ be $\{\neg\sigma: \sigma \in \Phi\}$. If the conclusion of (*)″ fails then clearly $T \cup (\neg)\Phi$ is consistent. But by applying 3.13, we obtain that $T \cup (\neg)\Phi$ has a model with CSS in A, contradicting $\mathrm{CSS}^A(T) \subset \Phi$.

Turning to (**), assuming (ii) we have by 3.17 (or 2.15) that $\mathrm{CSS}^A(T)$ is a subset of some A-finite set iff it is A-finite. Also, we can prove the following generaliza tion of (**):

(**)′ For any A-scattered K defined by $\exists(L' - L) \wedge \Sigma$, with Σ a Σ definable theory in L'_A, if $\mathrm{CSS}^A(K)$ is not a subset of any A-finite set, then K has a member with canonical Scott sentence not in A. The proof is obtained by noting that 3.10 implies that under the hypothesis, the theory

$$\Sigma \cup \{\neg\sigma_\alpha: \alpha < o(A)\}$$

is A-finitely consistent (for the σ_α, see the proof of 2.15). Hence, the assertion follows from the Barwise compactness theorem (and the meaning of the α_α).

Finally, we restate 3.10 to formulate our refinement of Morley's theorem 3.6.

Definition 3.19. For any sets a and b, we say that b is *strongly constructible* in a if b belongs to the smallest admissible set containing a and the set-theoretic rank of b, $r(b)$, as elements. (Clearly, "strongly constructible in" is stronger than "constructible in".)

Theorem 3.20. *For any Σ^1_1 sentence $\sigma = \exists\bar{R}\,\phi$ over $L_{\omega_1\omega}$, if the class K defined by σ is $\mathcal{L}^*[\sigma]$-scattered (in the sense of Definition 2.6; $\mathcal{L}^*[\sigma]$ is the class of sets strongly constructible in σ), then the canonical Scott sentence of every model in K is strongly constructible in σ.*

Proof. The assertion for countable models follows immediately from 3.10. The assertion in general follows by an easy application of Levy's absoluteness theorem: $\langle \mathrm{HC}, \in \restriction \mathrm{HC}\rangle$ is an elementary substructure of $\langle V, \in\rangle$ with respect to Σ formulas of set theory.

Remark 8. The scatteredness of a class K in Morley's sense, with K defined by a Σ_1^1 sentence σ over $L_{\omega_1\omega}$, can be expressed equivalently by saying that the structure $\langle TC(\sigma) \cup \omega, \varepsilon \restriction TC(\sigma) \cup \omega \rangle$ satisfies a certain Π_2^1 second order sentence. It follows from Shoenfield's absoluteness theorem that being scattered is absolute with respect to suitable, e.g. ordinary Boolean, extensions. It can easily be derived from this and Morley's theorem 3.D that K as above is scattered iff in all Boolean extensions V' of the universe, K has at most $\aleph_1$ non-isomorphic countable members, and also, iff there is such V' such that K has $< 2^{\aleph_0}$ countable members in V'.

Finally, on open problems, we have to say that there are altogether too many of them. Of course, the main one is Vaught's conjecture: If the class defined by a sentence in $L_{\omega_1\omega}$ is scattered, it has only countably many countable members.

4. Constructing models of power $\aleph_1$

In this section we will apply 3.8 in conjunction with some recent work of J.-P. Ressayre's [24] to prove the existence of certain models of power $\aleph_1$. The main role in these proofs is played by Ressayre's work; our 3.8 will enter only at a single point, viz. in Lemma 4.3.

We will quote as much as we need from Ressayre's very important paper [24]. We will consider countable admissible sets A containing ω as an element, which have the form $\mathcal{L}_\alpha[a]$. For an arbitrary $b \in HC$, and $A = \mathcal{L}_\alpha[a]$ we write $A[b]$ for $\mathcal{L}_\alpha[\langle a, b\rangle]$. We note that $A[b]$ is admissible iff there is an admissible $B \supseteq A$ such that $o(B) = o(A)$ and $b \in B$.

Definition 4.1. (Ressayre [24]). Let A be as above, $\omega \in A$, L a language, $L \in A$. We call an L-structure $\mathcal{M}$ L_A*-saturated* if there is an isomorphic copy $\mathcal{M}'$ of $\mathcal{M}$ such that $|\mathcal{M}'| = \omega$ and $A[\mathcal{M}']$ is admissible.

Remark 1. Ressayre also gives an equivalent definition, in fact for arbitrary countable admissible A, that indeed justifies the name for the notion.

Lemma 4.2. (Ressayre [24]). (i) *The union of a countable* L_A*-elementary chain of* L_A*-saturated models is again* L_A*-saturated.*

(ii) *Let's call a model* $\mathcal{M}$ L_A*-extendible (or simply, extendible) if it is* L_A *saturated and there is an* L_A*-saturated* $\mathcal{N}$ *such that* $\mathcal{M} \precneqq_{L_A} \mathcal{N}$. *If a* Σ*-theory* T *in* L_A *has a pair of models* $\mathcal{M}_0 \precneqq_{L_A} \mathcal{N}_0$, *then it has an extendible model.*

(iii) *Any* L_A*-saturated model which is* L_A*-equivalent to an extendible model is itself extendible. As a consequence,*

(iv) *For any extendible* $\mathcal{M}$, *there is an* L_A*-elementary chain* $\langle \mathcal{M}_\alpha : \alpha < \omega_1 \rangle$ *of extendible models* $\mathcal{M}_\alpha$ *such that* $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_\alpha \precneqq_{L_A} \mathcal{M}_{\alpha+1}$ *and* $\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_{\alpha+1}$ $(\alpha, \lambda < \omega_1)$. *We call* $\langle \mathcal{M}_\alpha : \alpha < \omega_1 \rangle$ *a chain associated with* $\mathcal{M}$.

Remark 2. (ii) and (iii) are derived in [24] from much more general results. The lemma constitutes the outline of Ressayre's proof of a theorem originally proved by J. Gregory [6]: if T is as in (ii), T has an uncountable model.

Recall the notion of an L_A-homogeneous model, called (ω, L_A)-homogeneous in [10]. A countable model $\mathcal{M}$ is L_A-*homogeneous* if for any $n < \omega$ and any n-tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ of elements of $|\mathcal{M}|$, if $(\mathcal{M}, \boldsymbol{a}) \equiv_{L_A} (\mathcal{M}, \boldsymbol{b})$, then $(\mathcal{M}, \boldsymbol{a})$ and $(\mathcal{M}, \boldsymbol{b})$ are isomorphic. The following Lemma 4.3 contains our application of 3.8.

In the Lemma and elsewhere in this section we will refer to PC_A classes meaning a class K defined by $\exists(L' - L) \wedge T$ for some L'_A theory T Σ-definable on A, with some A-finite L.

Lemma 4.3. *Let the* L_A*-saturated model* $\mathcal{M}$ *belong to some* A*-scattered* PC_A*-class. Then* $\mathcal{M}$ *is* L_A*-homogeneous.*

Proof. Let $\phi^{\alpha}_{\boldsymbol{a}}(\boldsymbol{x})$ be the canonical α-type of $\boldsymbol{a} \in {}^{\omega}|\mathcal{M}|$ in $\mathcal{M}$. Recall Nadel's theorem (Theorem 1.3 in [22], cf. also [21]): if $\mathcal{M} \in B$, B is admissible and $\mathcal{M} \models \phi^{o(B)}_{\boldsymbol{a}}[\boldsymbol{b}]$, then $(\mathcal{M}, \boldsymbol{a}) \cong (\mathcal{M}, \boldsymbol{b})$. Now we have an admissible B such that $o(B) = o(A)_{\mathrm{df}} = \alpha_0$ and $\mathcal{M} \in B$. Also, since $\mathcal{M}$ is a member of an A-scattered PC_A-class, by 3.8 we have that each $\phi^{\alpha}_{a}(\boldsymbol{a} \in {}^{\omega}|\mathcal{M}|,\ \alpha < \alpha_0)$ is a formula of the fragment L_A. Notice that $\phi^{o(B)}_{\boldsymbol{a}}$ is the conjunction of all $\phi^{\alpha}_{\boldsymbol{a}}$ for $\alpha < \alpha_0$. Hence, if $(\mathcal{M}, \boldsymbol{a}) \equiv_{L_A} (\mathcal{M}, \boldsymbol{b})$ then $\mathcal{M} \models \phi^{o(B)}_{\boldsymbol{a}}[\boldsymbol{b}]$. By Nadel's theorem, the desired conclusion follows.

Remark 3. We want to point out that the application of 3.8 in the above proof is not redundant, i.e. that not all L_A-saturated models are L_A-homogeneous. Consider the language L consisting of the binary relation symbols R_n, $n < \omega$. It is straightforward to construct an L-model $\mathcal{M}$ of power $2^{(2^{\aleph_0})}$ such that for any $B \subset {}^{\omega}2$ there is $a_B \in |\mathcal{M}|$ such that

$$\{f \in {}^{\omega}2 \colon \exists b\, \forall n\, [f(n) = 1 \Leftrightarrow \mathcal{M} \models R_n a_B b]\} = B.$$

It follows that for

$$\sigma(u, v) =_{\mathrm{df}} \forall x\, \exists y \bigwedge_{n<\omega} [R_n ux \leftrightarrow R_n vy]$$

$$\wedge\, \forall y\, \exists x \bigwedge_{n<\omega} [R_n ux \leftrightarrow R_n vy]$$

we have that

$$\mathcal{M} \models \sigma[a_B, a_{B'}] \text{ implies that } B = B'.$$

Since $2^{\aleph_0} < 2^{(2^{\aleph_0})}$, there are $B \neq B'$ such that a_B and $a_{B'}$ have the same L_A-types in $\mathcal{M}$.

We have shown that the following A-recursive theory

$$\{\phi(c) \Leftrightarrow \phi(d) \colon \phi(x) \in L_A\} \cup \{\neg\, \sigma(c, d)\}$$

(with two distinct new constants c, d) is consistent. By the general existence theorem for L_A-saturated models (cf. [24], and in a disguised form in [5]), this theory has an L_A-saturated model $(\mathcal{M}, c, d)$. But then $\mathcal{M}$ is L_A-saturated and it cannot be L_A-homogeneous.

Lemma 4.4. *Let $\mathcal{M}$ be an extendible model and $\langle \mathcal{M}_\alpha : \alpha < \omega_1 \rangle$ an associated chain as in 4.2(iv). Let K be a PC_A-class which is scattered with respect to L_A (i.e., $S_n^{L_A}(K)$ is countable for each $n < \omega$). Assume that each $\mathcal{M}_\alpha (\alpha < \omega_1)$ belongs to K. Let $\mathcal{N} = \bigcup_{\alpha < \omega_1} \mathcal{M}_\alpha$. Then there is $\alpha < \omega_1$ such that*

$$\mathcal{M}_\alpha \equiv_{\infty\omega} \mathcal{N}.$$

Proof. Let S_α be the set of all L_A-types realized by finite tuples of elements in $\mathcal{M}_\alpha$. Then $S_\alpha \subseteq S_\beta$ for $\alpha \leqslant \beta < \omega_1$ and each S_α is a subset of the countable set $S = \bigcup_{n<\omega} S_n^{L_A}(K)$. Hence there is $\alpha < \omega_1$ such that $S_\beta = S_\alpha$ for $\beta \geqslant \alpha$. We claim that $\mathcal{M}_\alpha \equiv_{\infty\omega} \mathcal{N}$. In fact, the relation $\boldsymbol{a} \sim \boldsymbol{b}$ defined for $\boldsymbol{a} \in^\omega |\mathcal{M}_\alpha|$, $\boldsymbol{b} \in^\omega |\mathcal{N}|$ by

$$\boldsymbol{a} \sim \boldsymbol{b} \Leftrightarrow_{\mathrm{df}} (\mathcal{M}_\alpha, \boldsymbol{a}) \equiv_{L_A} (\mathcal{N}, \boldsymbol{b})$$

has the back-and-forth property (cf. e.g. [1]). To show this, assume that $\boldsymbol{a} \sim \boldsymbol{b}$ and let b be an arbitrary element of $|\mathcal{N}|$. Then $b \in |\mathcal{M}_\beta|$ for some $\beta \geqslant \alpha$. From $S_\beta = S_\alpha$ it follows that there is a finite tuple $\boldsymbol{c} {}^\frown c$ in $|\mathcal{M}_\alpha|$ such that

$$(\mathcal{M}_\alpha, \boldsymbol{c}, c) \equiv_{L_A} (\mathcal{M}_\beta, \boldsymbol{b}, b) \equiv_{L_A} (\mathcal{N}, \boldsymbol{b}, b).$$

In particular,

$$(\mathcal{M}_\alpha, \boldsymbol{a}) \equiv_{L_A} (\mathcal{M}_\alpha, \boldsymbol{c}).$$

Since K is scattered with respect to L_A, it is a fortiori A-scattered. By 4.3, $\mathcal{M}_\alpha$ is L_A-homogeneous. Hence there is $a \in |\mathcal{M}_\alpha|$ such that

$$(\mathcal{M}_\alpha, \boldsymbol{a}, a) \equiv_{L_A} (\mathcal{M}_\alpha, \boldsymbol{c}, c).$$

It follows that

$$\boldsymbol{a} {}^\frown a \sim \boldsymbol{b} {}^\frown b$$

as desired. The dual back-and-forth condition is shown similarly (and more simply).

This completes the proof of the lemma.

Theorem 4.5. *Any counterexample to Vaught's conjecture has a countable model which is ∞, ω-equivalent to an uncountable model. More precisely, if T is a Σ-definable theory on A, $L \in A$, $T \subset L_A$, T is scattered with respect to L_A and it has some model $\mathcal{M}$ with Scott height $> o(A)$, then it has models $\mathcal{M}$ and $\mathcal{N}$ such that $\|\mathcal{M}\| = \aleph_0$, $\|\mathcal{N}\| = \aleph_1$, the Scott height of $\mathcal{M}$ is $\leqslant o(A)$ and $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$.*

Proof. Consider any L_A-complete extension T′ of T. The prime model of T′ has Scott height $\leqslant o(A)$ (cf. 3.12). Hence by the assumption, there is a completion T′

which is not $\aleph_0$-categorical. If $\mathcal{N}_0$ is the prime model of T′ and $\mathcal{N}_1$ is any other model of T′, then $\mathcal{N}_0$ is L_A-elementarily embeddable into $\mathcal{N}_1$, i.e., we can assume that $\mathcal{N}_0 <_{L_A} \mathcal{N}_1$. $\mathcal{N}_1$ can be taken non-isomorphic to $\mathcal{N}_0$, hence $\mathcal{N}_0 \lneqq_{L_A} \mathcal{N}_1$. Now we apply 4.2(ii) and (iv). We have an extendible model $\mathcal{M}_0$ of T with an associated chain $\langle \mathcal{M}_\alpha : \alpha < \omega_1 \rangle$. Let $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. All $\mathcal{M}_\alpha$ belong to Mod(T), a class scattered with respect to L_A. Hence by 4.4 there is $\alpha < \omega_1$ such that $\mathcal{M} =_{df} \mathcal{M}_\alpha \equiv_{\infty,\omega} \mathcal{N}$. Since $\mathcal{M}_\alpha$ is L_A-saturated, an isomorphic copy of it belongs to an admissible set B with $o(B) = o(A)$, hence by the theorem of Nadel [22] quoted in the proof of 4.3, the Scott height of $\mathcal{M}$ is $\leqslant o(A)$. Q.E.D.

Remark 3. Although we gave the definition of A-saturatedness only for certain special A, Ressayre's original definition and all the proofs above work for an arbitrary countable admissible A.

Remark 4. Theorems 4.5 and 3.3 have a common but still significant special case. This is the case of a class K = Mod(T) with a Σ-theory T such that K has $< 2^{\aleph_0}$ countable models and each countable model in K has only countably many automorphisms. By "Kueker's Corollary" (cf. [11]), no countable model in K can have an uncountable ∞, ω-equivalent. Hence, by 4.5 we obtain that T satisfies Vaught's conjecture. We don't know if every absolutely characterizable countable $\mathcal{M}$ (i.e., such that there is no uncountable $\mathcal{N} \equiv_{\infty\omega} \mathcal{M}$) is also invariantly characterizable (cf. [21], i.e. $\mathrm{CSS}(\mathcal{M}) \in \mathcal{M}^+$). If the answer is "yes", then 4.5 is a consequence of 3.3.

Remark 5. Theorem 4.5 says in particular that a counterexample to Vaught's conjecture has models of power $\aleph_1$ and in fact, the proof shows that it has at least $\aleph_1$ ones. Professor L. Harrington recently showed us how to prove that in fact there are $\aleph_2$ non-isomorphic models of power $\aleph_1$ for such theories. His proof gives models that are not ∞, ω-equivalent to countable ones. Independently of this, with V. Harnik we have found another result (cf. [8]) that has the corollary that a counterexample has at least one uncountable model that is not ∞, ω-equivalent to any countable model.

For the purposes of our last result we need the following

Lemma 4.6. *Let* $A = a^+ = \mathcal{L}_{\omega_1^a}[a]$ *be the next admissible set to* a, $a \in \mathrm{HC}$, $\mathrm{L} \in A$. *Then the class of* L_A*-saturated models of type* L *and the class of extendible models of type* L *are both* PC_A *classes of countable models (i.e., they coincide with the class of the countable models in a* PC_A *class).*

Proof. We consider only infinite models. Consider the following theory in an extended language of set-theory.

KP,

diag(A),

"$\boldsymbol{M}$ is an L structure, $|\boldsymbol{M}| = \omega$".

$(\forall \alpha \in \mathrm{ord})$ "$\mathscr{L}_\alpha[a]$ is not a model of KP".

For any normal $\mathcal{N} \models \mathrm{T}$, $\mathcal{M} =_{\mathrm{df}} \boldsymbol{M}^{\mathcal{N}}$ is an actual L-structure. We claim that a countable L-structure $\mathcal{M}$ is L_A-saturated iff it is isomorphic to $\boldsymbol{M}^{\mathcal{N}}$ for some $\mathcal{N} \models \mathrm{T}$. If $\mathcal{M}$ is A-saturated, we can take an isomorphic copy $\mathcal{M}'$ with $|\mathcal{M}'| = \omega$ and the admissible set $A[\mathcal{M}']$ such that $\mathcal{N} = \langle A[\mathcal{M}'], \in \restriction A[\mathcal{M}'], c, \mathcal{M}'\rangle_{c \in A} \models \mathrm{T}$ (note that $A = a^+!$) and $\mathcal{M}' = \boldsymbol{M}^{\mathcal{N}}$. Conversely, if $\mathcal{N} \models \mathrm{T}$, then the standard part of $\mathcal{N}$, WF($\mathcal{N}$), is an admissible set, by the last sentence in T it does not contain more ordinals than A, and $\mathcal{M}' =_{\mathrm{df}} \boldsymbol{M}^{\mathcal{N}} \in \mathrm{WF}(\mathcal{N})$. It follows that $\mathcal{M}'$ and any $\mathcal{M}$ isomorphic to $\mathcal{M}'$ are A-saturated, showing the claim.

Consider the following theory T′ in a langage $\mathrm{L}' \supset \mathrm{L}$:

T,

"$\boldsymbol{F}$ is an isomorphism of the L-reduct of the model to $\boldsymbol{M}$".

Then T′ is Δ-definable on A and $\exists(\mathrm{L}' - \mathrm{L}) \wedge \mathrm{T}'$ defines the class of A-saturated models.

Let us duplicate every element R of L to $\boldsymbol{R}$. Let $\mathbf{L} = \{\boldsymbol{R} : R \in \mathrm{L}\}$ and let $\boldsymbol{\phi}$ denote the result of replacing each $\mathrm{R} \in \mathrm{L}$ by $\boldsymbol{R}$. Consider the theory T″ in language L″, defined as follows:

T′,

$\mathbf{T}'$ [which is the theory ensuring that the $\mathbf{L}$-reduct of the model is $\mathbf{L}_A$-saturated]

"G is a 1–1 function with a range which is a proper subset of the universe",

$\forall x_1 \cdots x_n [\phi(x_1, \ldots, x_n) \Leftrightarrow \boldsymbol{\phi}(Gx_1, \ldots, Gx_n)]$ for each $\phi \in \mathrm{L}_A$.

Then T″ is Δ-definable on A and $\exists(\mathrm{L}'' - \mathrm{L}) \wedge \mathrm{T}''$ defines the class of extendible models. Q.E.D.

Theorem 4.7. *Let σ be a Σ^1_1 sentence over $\mathrm{L}_{\omega_1\omega}$, and assume that σ has at least one, but less than $2^{\aleph_0}$ non-isomorphic models of power $\aleph_1$. Then σ has models $\mathcal{M}$ and $\mathcal{N}$ such that $\|\mathcal{M}\| = \aleph_0$, $\|\mathcal{N}\| = \aleph_1$ and $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$. Moreover, $\mathcal{M}$ can be chosen so that $|\mathcal{M}| = \omega$ and $\omega_1^{\langle\sigma, \mathcal{M}\rangle} = \omega_1^\sigma$ (and a fortiori, the Scott height of $\mathcal{M}$ is $\leq \omega_1^\sigma$).*

Proof. Recall Keisler's following theorem ([10], Theorem 45): If σ has an uncountable model which realizes uncountably many types over a countable fragment of $\mathrm{L}_{\omega_1\omega}$, then σ has $2^{\aleph_1}$ non-isomorphic models of power $\aleph_1$. It follows that every model of σ of power $\aleph_1$ realizes only countably many L_A-types for any countable fragment L_A.

Now let $\sigma = \exists \vec{R}\phi'$, $\phi' \in L'_{\omega_1\omega}$, $A = \sigma^+$, let K be the class of L'_A-extendible models of ϕ' and let $K = K' \upharpoonright L$. By 4.2(iv), every $\mathcal{M}' \in K'$ can be L'_A-elementarily extended to a model of ϕ' of power $\aleph_1$. Hence every type in $S = \bigcup_{n<\omega} S_n^{L_A}(K)$ is realized in some model of σ of power $\aleph_1$. Since there are $< 2^{\aleph_0}$ such models and each realizes only countably many L_A-types, we have that $|S| < 2^{\aleph_0}$. In other words, K is scattered with respect to L_A.

By 4.6, K' and hence K too, are PC_A-classes. By 4.2(ii) and the fact that ϕ' has at least one uncountable model, it is easy to see that K is non-empty. Take any $\mathcal{M}'_0 \in K'$ and by 4.2(iv), some chain $\langle \mathcal{M}'_\alpha : \alpha < \omega_1 \rangle$ associated with $\mathcal{M}'_0$. Let $\mathcal{M}_\alpha = \mathcal{M}'_\alpha \upharpoonright L$. Then the chain $\langle \mathcal{M}_\alpha : \alpha < \omega_1 \rangle$ is associated with $\mathcal{M}_0$ in the sense of 4.2(iv). Since K is an A-scattered PC_A class, by 4.4 for some α, $\mathcal{M} =_{df} \mathcal{M}_\alpha$ and $\mathcal{N} =_{df} \bigcup_{\beta<\omega_1} \mathcal{M}_\beta$ will satisfy the theorem, since $\mathcal{N} = \mathcal{N}' \upharpoonright L$ where $\mathcal{N}' = \bigcup_{\beta<\omega_1} \mathcal{M}'_\beta$.

The following is a special case of 4.7.

Corollary 4.8. *If the Σ^1_1 sentence σ is $\aleph_1$-categorical, then the model of σ having power $\aleph_1$ is ∞, ω-equivalent to a countable model $\mathcal{M}$. $\mathcal{M}$ can be found such that* $|\mathcal{M}| = \omega_1^{(\mathcal{M}, \sigma)} = \omega_1^\sigma$.

Remark 6. In [19], a theorem of S. Shelah's is announced that draws a stronger conclusion (without the last sentence of 4.8) from the assumption that $\sigma \in L_{\omega_1\omega}$, σ is $\aleph_1$-categorical and has arbitrarily large models. If in 4.8 we assume that σ has arbitrarily large models, then the conclusion of 4.8 without the last sentence follows from Lemma E, p. 97, in [10].

Added September 8, 1975

In [27], Shelah proves 4.7 and 4.8, actually in a stronger form: in 4.7, he needs only $2^{\aleph_1}$ in place of $2^{\aleph_0}$. Most importantly, he completely avoids any tool like our Main Lemma. Although he does not talk about admissible sets, the additional information in 4.7 and 4.8 can be recovered easily from his work. The only information we could find in our treatment that perhaps does not come out of his proof is that, in 4.8, any L_A-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$, an uncountable model of σ, is actually $L_{\infty\omega}$-elementary. Although [27] was written after the present paper, Professor Shelah informs me that the theorem in question had been known to him before I knew it.

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